In addition to the vapour of the liquid, bubbles in cavitating flows usually contain also a certain amount of permanent gas that diffuses out of the liquid as they grow. This paper presents a simplified linear model for the propagation of monochromatic pressure waves in a bubbly liquid with these characteristics. Phase change effects are included in detail, while the gas is assumed to follow a polytropic law. It is shown that even a small amount of permanent gas can have a major effect on the behaviour of the system. Particular attention is paid to the low-frequency range, which is of special concern in flow cavitation. Numerical results for water and liquid oxygen illustrate the implications of the model.

1. Introduction

One of the important consequences of the presence of bubbles in cavitating flows is a marked reduction in the speed of propagation of pressure waves. This feature has a powerful feedback effect on the overall flow field and also on the dynamics of cavitation itself as, due to the low sound speed, phenomena associated with compressibility and shocks become possible even at relatively modest flow velocities. Pressure waves in the cavitating region are responsible for much of the unsteady features of cavitation. For example, the experiments of Reisman et al. [1] showed that condensation shocks cause large pressure spikes which influence the behaviour of sheet cavities (e.g. [2,3]), and recent work has demonstrated that condensation shocks are responsible for the transition from stable to cloud-shedding cavities [4].

In the numerical modelling of flow cavitation the pressure–density coupling plays a critical role in the propagation of information from the cavitating region to the rest of the domain (e.g. [5]) and, directly or indirectly, the speed of sound is involved in the modelling. For example, Delannoy & Kueny [6] introduced a relation of the form

\[ p = p_{\text{sat}} + \frac{1}{2} (\rho_v - \rho_s) c_{\text{min}}^2 \arcsin[(1 - 2\beta)] \]

for

\[ p_{\text{sat}} - \Delta p \leq p \leq p_{\text{sat}} + \Delta p, \]  

(1.1)

in which \( p \) is the mixture pressure, \( p_{\text{sat}} \) the saturated vapour pressure at the liquid temperature, \( \rho_v \) and \( \rho_s \) the saturated liquid and vapour densities, \( \beta \) the bubble volume fraction and \( \Delta p \) and \( c_{\text{min}} \) model parameters. The performance of this model which, as written in (1.1) or with some variations, has been used by several investigators (e.g. [7–11]), depends therefore very strongly on the determination of the parameter \( c_{\text{min}} \) which represents the minimum speed of pressure waves in the mixture. No consensus currently exists on the correct value of \( c_{\text{min}} \) or, more generally, on the dependence of the speed of sound on bubble volume fraction, composition, frequency and others (e.g. [12]).

There is abundant experimental information (reviewed e.g. in [13]) on the speed of sound in a liquid containing bubbles with a negligible vapour content. On the other hand, experimental information on the speed of sound in a bubbly liquid where the bubbles contain an appreciable amount of vapour is very sparse at best. The latest study is a paper by Shamsborhan et al. [12] which...
is, however, somewhat inconclusive. Surprisingly (but see below in §4.1), the dependence of the data on the bubble volume fraction is in reasonable agreement with the classic Wood formula [14,15] which, in principle, should only apply to bubbles containing a permanent gas rather than a vapour. Furthermore, even though frequencies from 0 to 2 kHz were used in the experiment, no frequency-dependent data were reported, although some frequency dependence would be expected on theoretical grounds.

These considerations illustrate the importance of a detailed analysis of the linear speed of sound in a bubbly mixture both to understand its physics and to provide a solid foundation for modelling. This is the object of the present paper, which differs from earlier studies in the explicit consideration of bubbles containing vapour in addition to a permanent gas.

This focus is motivated by the fact that, in typical cavitation phenomena, the liquid contains a non-negligible amount of dissolved permanent gas which diffuses into the bubbles as they expand and does not have the time to re-dissolve during collapse (e.g. [16,17]). A similar phenomenon occurs in boiling heat transfer in which the hot liquid tends to be supersaturated with dissolved air given the decreasing gas solubility with temperature (e.g. [18–20]). Instances occur also in cryogenic cavitation (e.g. in rocket engine pumps [21,22]) because gaseous cavitation, for example, is dissolved in liquid oxygen due to the proximity of the respective boiling points (87.35 K and 90.15 K, respectively) which makes it difficult to separate the two components in commercial oxygen liquefaction facilities. Similarly, gaseous helium (boiling point 4.15 K) is dissolved in liquid hydrogen (boiling point 20.28 K).

The present analysis is based on a relation which is well established in the case of uniformly distributed gas bubbles occupying a small fraction of the total volume, namely (e.g. [13,23])

\[
\frac{c_r^2}{c_m^2} = 1 + \frac{3\beta k_r^2}{R_0} \frac{1}{\omega_0^2 - \omega^2 + 2i\omega b},
\]

in which \(c_r\) and \(c_m\) are the linear speeds of sound in the pure liquid and in the mixture, respectively, \(\omega\) is the angular frequency of the pressure wave, \(\omega_0\) the effective resonance angular frequency of the bubbles and \(b\) the damping parameter of the bubble oscillations. For gas bubbles, the phase speed and attenuation predicted by this expression (which can readily be generalized to a mixture of bubble sizes as shown below) compare very well with data over a broad range of parameters (e.g. [13,24]) although, for reasons that are still unclear, the attenuation in the immediate neighbourhood of the bubble resonance frequency is somewhat over-estimated [25,26].

The main task that we address in this paper is the description of expression for \(\omega_0\) and \(b\) suitable for bubbles containing a mixture of vapour and permanent gas rather than permanent gas only as in the original theory. We provide derivations in the next section. Section 3 illustrates the consequences of the theory for the case of individual bubbles while the results for \(c_m\) are addressed in §4.

2. Analysis

In this section we provide a concise derivation of (1.2) and then proceed to the calculation of the effective resonance frequency and damping of a bubble containing a mixture of vapour and a permanent gas. The qualifier ‘effective’ is used because, as will be seen, the apparent stiffness and dissipation of a gas–vapour bubble depend on the driving frequency: true resonance only occurs when the driving frequency happens to coincide with the effective resonance frequency. Some details on the calculations can be found in the electronic supplementary material.

2.1. Speed of sound in a dilute bubbly liquid

A straightforward derivation of the relation (1.2) for the pressure-wave speed, or speed of sound, can be based on the expression for the average density \(\rho_m\) of the mixture:

\[
\rho_m = (1 - \beta)\rho_1 + \beta \rho_b,
\]

where \(\rho_1\) and \(\rho_b\) are the densities of the liquid and bubble contents, respectively, the latter being a mixture of vapour and permanent gas. We assume that the bubbly liquid is spatially uniform. In the homogeneous flow model in which the bubbles move with the liquid, the pressure-wave speed is found by differentiating this expression with respect to the mixture pressure \(p\):

\[
\frac{1}{c_m^2} = \frac{\rho_m}{\rho_1} \frac{\partial p}{\partial p} = \frac{1 - \beta}{\rho_1} + \frac{1}{\rho_b} \left( \frac{\beta}{\rho_1} - \frac{\beta^2}{\rho_b} \right) \frac{\partial \rho}{\partial p},
\]

with \(c_0\) the speed of sound of the gas–vapour mixture in the bubbles. Let us consider a volume \(V = V_1 + N \rho_b\), of the mixture consisting of a volume \(V_1\) of liquid and \(N\) bubbles of equal undisturbed volume \(\rho_b\).

By definition \(\beta = N \rho_b/(V_1 + N \rho_b)\) and, therefore,

\[
\frac{d\beta}{dp} = \frac{1}{V} \left( \frac{\beta}{\rho_0} \frac{d \rho}{dp} \right) + \frac{1}{V_1} \frac{d V_1}{dp}.
\]

In carrying out this calculation we have used the fact that, for bubbles moving with the liquid, neither \(N\) nor \(V_1\) change in the course of the propagation of the wave (e.g. [23,27]). It is known that bubbles–liquid relative motion only introduces a correction of order \(\beta\) [23]; thus our assumption of constant \(N\) is not likely to introduce strong limitations in the validity of the present theory.

Upon substituting into (2.2) we find

\[
\frac{1}{c_m^2} = \left( \frac{\beta}{\rho_1} \frac{\beta}{\rho_b} \frac{\beta}{\rho_1} - \frac{1}{\rho_1} \right) \frac{1}{\rho_1} \left( \frac{\beta}{\rho_1} - \frac{1}{\rho_1} \right) + \frac{1}{\rho_1} \left( \frac{\beta}{\rho_1} - \frac{\beta}{\rho_1} \right),
\]

The theory is applicable only for small bubble volume fractions, so that all terms containing \(\beta^2\) can be dropped. Furthermore, \(\rho_1/\rho_b \ll 1\), \(\beta = 1/\beta^2 \approx 1/\epsilon^2\) and \(\beta^2/c_0^2\) also gives a negligible contribution. Thus, this relation can be simplified to

\[
\frac{1}{c_m^2} \approx \frac{1}{c_r^2} + \frac{3 \beta d R}{R_0},
\]

with \(R = R(t)\) the instantaneous radius of the bubbles, all having the equilibrium radius \(R_0\). Upon setting \(R(t) = R_0(1 + X(t))\), it is found that, in a sinusoidal pressure field with angular frequency \(\omega\), \(X(t)\) satisfies an equation of the form (see the next section)

\[
\ddot{X} + 2b\dot{X} + \omega_0^2 X = \frac{P_{ac}}{\rho_0 R_0^2} e^{i\omega t},
\]

with \(P_{ac}\) the acoustic pressure amplitude. As, in a linear theory,
we can set \(dR/dp = (dR/dt)/(dp/dt)\), it is readily seen that (2.5) reduces to (1.2).

The previous derivation is readily extended to a liquid containing bubbles of different sizes. If \(dN(R_0) = f(R_0)dR_0\) is the number of bubbles per unit volume with equilibrium radius between \(R_0\) and \(R_0 + dR_0\), the result is (e.g. [13])

\[
\frac{c_v^2}{\rho c} = 1 + 4\pi \frac{a^2}{R^2} \int_0^\infty f(R_0) R_0 \frac{dR_0}{R_0^2 - a^2 + 2ib\omega},
\]

(2.7)
in which both \(a_0\) and \(b\) depend now on \(R_0\), in addition to \(\omega\).

### 2.2. Harmonic oscillations of a bubble containing a mixture of vapour and gas

We base our model for bubble oscillations on the Rayleigh–Plesset equation (e.g. [28]):

\[
\frac{dR}{dt} + \frac{3}{2} R^2 = \frac{1}{\rho (t)} \left( p_v - p_g + \frac{2\sigma}{R} - 4\mu c \frac{R^2}{R} - p_{\infty} \right),
\]

(2.8)
in which \(\mu c\) is the liquid viscosity, \(\sigma\) the interfacial tension and \(P_{\infty}(t)\) the time-dependent pressure in the liquid. We have assumed that the bubble internal pressure is simply given by the sum of the partial pressures of the permanent gas, \(p_g\), and of the vapour, \(p_v\). Kumar & Brennen [29] have pointed out limitations of this model for large-amplitude volume changes of the bubble, which are beyond the linear regime of present interest. A greater concern for linear oscillations, particularly at high frequency, is the compressibility of the liquid and the associated radiation energy losses which are not included in (2.8); we will simply add the corresponding acoustic damping at the end of the derivation, a procedure justified by the linear nature of the present theory. The Rayleigh–Plesset equation is a consequence of the momentum balance for the liquid and it is therefore applicable also in the presence of strong thermal effects provided the bubble internal pressure is correctly calculated accounting for them.

We set

\[ R = R_0[1 + X(t)] \quad \text{and} \quad P_{\infty} = p_{\infty} + p_{\infty}e^{i\omega t} \]

and

\[ p_g = p_{g0} + q_g \quad \text{and} \quad p_v = p_{sat} + q_v, \]

(2.10)

with

\[ p_{g0} + p_{sat} = p_{\infty} + \frac{2\sigma}{R_0}, \]

(2.11)

Here \(p_{g0}\) is the partial pressure of the permanent gas in the bubble at equilibrium and \(p_{\infty}\) the undisturbed ambient pressure. After linearization we are then led to

\[
\dot{X} = \frac{1}{\rho c R_0^3} \left[ q_g + q_v + \frac{2\sigma R_0}{X - 4\mu c X} - p_{\infty}e^{i\omega t} \right].
\]

(2.12)

It is now necessary to relate \(q_{g}\) and \(q_{v}\) to \(X\) so as to cast this equation in the form (2.6) and identify the expressions of \(a_0\) and \(b\) suitable for the present case.

For the former quantity we use a simple polytropic relation \(p_{g0}R^\kappa = \text{const.}\) to find

\[ q_{g} = -3\kappa p_{g0}X, \]

(2.13)
in which \(\kappa\) is a polytropic index (e.g. [28]). The calculation of \(q_{v}\) is somewhat more involved. We start from the Clausius–Clapeyron relation (written with the approximation \(p_c \ll p_{\infty}\)):

\[
p_{sat} + q_v = p_{sat} + \frac{dp}{dT}(T_s - T_\infty)
\]

\[ = p_{sat} + \frac{h_v}{T_\infty}(T_s - T_\infty),\]

(2.14)
in which \(T_s\) is the liquid temperature at the bubble surface, \(T_\infty\) and \(p_c\) are the undisturbed liquid temperature and saturated vapour density and \(h_v\) is the latent heat of vaporization. The temperature distribution in the liquid is found from the liquid energy equation in the form

\[ T - T_\infty = \frac{R_0}{T_s}(T_s - T_\infty) \exp \left[ -\frac{h_v}{\sqrt{D_c}(r - R_0)} \right].\]

(2.15)

In deriving this result we have assumed that the liquid region thermally influenced by the bubble, which has dimensions of the order of the diffusion length \(\sqrt{D_c/\omega}\), is well separated from that of neighbouring bubbles. This assumption introduces a significant limitation at low frequencies as will be discussed in detail in §5.

If the vapour pressure and density in the bubble are taken as approximately uniform, \(T_s\) can be related to the other quantities from the linearized form of the energy balance for the bubble (e.g. [30])

\[ 4\pi R^2 k_r T_{\infty} = \frac{R}{\rho c_p} \frac{dT}{dt} \]

\[ + 4\pi R^3 k_c c_s \frac{dT}{dt}, \]

(2.16)
in which \(k_r\) is the liquid thermal conductivity and \(c_s = c_p - h_v/T_s\) with \(c_p\) the vapour specific heat at constant pressure, is the vapour specific heat along the saturation line (e.g. [31, §82]). The final result is

\[ q_v = -\frac{h_v^2 R_0^2}{\rho c_p D_c T_\infty} \]

\[ \times \sqrt{\frac{\omega R_0^2}{2D_c} + 1} \left( \frac{\omega R_0^2}{2D_c} + \left( \frac{1}{3} \frac{\omega R_0^2}{2D_c} + \frac{1}{3} \frac{\omega R_0^2}{2D_c} \frac{\rho c_p}{\rho c_p} \right) ^2 \right) \]

\[ \left( \frac{\omega R_0^2}{2D_c} + 1 \right)^2 - \frac{\omega R_0^2}{2D_c} \left( 1 + \frac{2}{3} \sqrt{\frac{\omega R_0^2}{2D_c} \frac{\rho c_p}{\rho c_p}} \right) \]

\[ \left[ \frac{\omega R_0^2}{2D_c} + \frac{1}{3} \frac{\omega R_0^2}{2D_c} \frac{\rho c_p}{\rho c_p} \right] ^2 \]

\[ \times \frac{2\pi R_0^2 \rho c_p}{\rho c_p D_c T_\infty} \]

\[ \times \left( \frac{\omega R_0^2}{2D_c} + 1 \right)^2 \left( \frac{\omega R_0^2}{2D_c} + \left( \frac{1}{3} \frac{\omega R_0^2}{2D_c} \frac{\rho c_p}{\rho c_p} \right) ^2 \right) \]

\[ \left( \frac{\omega R_0^2}{2D_c} + 1 \right)^2 - \frac{\omega R_0^2}{2D_c} \left( 1 + \frac{2}{3} \sqrt{\frac{\omega R_0^2}{2D_c} \frac{\rho c_p}{\rho c_p}} \right) \]

\[ \times \frac{2\pi R_0^2 \rho c_p}{\rho c_p D_c T_\infty} \]

\[ \times \left( \frac{\omega R_0^2}{2D_c} + 1 \right)^2 \left( \frac{\omega R_0^2}{2D_c} + \left( \frac{1}{3} \frac{\omega R_0^2}{2D_c} \frac{\rho c_p}{\rho c_p} \right) ^2 \right) \]

(2.19)

and

\[ D = \frac{h_v^2 R_0^2}{\rho c_p D_c T_\infty} \]

\[ \times \frac{2\pi R_0^2 \rho c_p}{\rho c_p D_c T_\infty} \]

\[ \times \left( \frac{\omega R_0^2}{2D_c} + 1 \right)^2 \left( \frac{\omega R_0^2}{2D_c} + \left( \frac{1}{3} \frac{\omega R_0^2}{2D_c} \frac{\rho c_p}{\rho c_p} \right) ^2 \right) \]

(2.20)
In the previous derivation we have neglected the mutual diffusion of the gas and vapour in the bubble. The hindrance to vapour condensation offered by the presence of a gas-rich layer adjacent to the liquid surface is a well-known phenomenon in condensation heat transfer (e.g. [32]) which has been considered, in the specific context of a gas–vapour bubble, in Khabeev [33]. The effect becomes notable at frequencies above about $D_b/\pi R_0^2$, where $D_b$ is the gas–vapour mass diffusivity. For a 100 µm radius bubble and typical values of $D_b$ this limit is around 1 kHz. A reduction in the condensation rate due to hindered diffusion may be expected to increase somewhat the already strong effects of a permanent gas due to an increase in the ‘stiffness’ of the bubble. Solubility of the gas in the liquid has also been considered in Al-Monnai & Khabeev [34] but it has not been found to be significant for sparingly soluble gases such as air and even CO₂ in water. The gas–vapour inter-diffusion effect has also been included in the model of Sochard et al. [35] who however were interested in the case of a relatively cold liquid, small vapour content and large-amplitude bubble oscillations.

3. Results for single bubbles

Upon substitution of (2.13) and (2.18) into (2.12) we find an equation of the form (2.6) with

$$\omega_0^2 = \frac{1}{\rho_l R_0^3} \left( S + 3k \rho_l g_0 - \frac{2\alpha}{K_0} \right)$$

and

$$2b = D + \frac{4\mu_g}{\rho_l R_0^3} + \frac{\alpha^2 R_0}{c_r}$$

in which $S$ and $D$ are given by (2.19) and (2.20) and we have added the last term in the expression of $b$ to account for the radiation damping caused by the compressibility of the liquid (e.g. [36–38]). For a bubble containing only gas, $S = D = 0$ and (3.1) reduces to the well-known expression for the resonance frequency of a gas bubble in the polytropic approximation while (3.2) becomes the sum of the viscous and acoustic damping parameters (e.g. [28,36,37]). Due to the use of the polytropic form for $\rho_{0g}$, the damping due to thermal effects of the gas is not recovered in this approximation but, here, our focus is on vapour effects which dominate the damping in the situations of present concern. If desired, the thermal damping can be simply added similarly to the acoustic contribution.

The ratio $\rho_l c_r/(\rho_{0g} c_p)$ appearing in the expressions for $S$ and $D$ is typically small. For water at 10°C and 100°C, for example, it has the values $-8.2 \times 10^{-6}$ and $-6.1 \times 10^{-4}$, while for oxygen at the normal melting and boiling points, 54.37 K and 90.15 K, it has the values $-1.6 \times 10^{-3}$ and $-3.3 \times 10^{-3}$. The product of this ratio by $R_0 \sqrt{\omega/2D_c}$ would be important for large bubbles or high frequencies, both cases in which our assumption of spatial uniformity of the bubble interior may well fail. Thus, we feel justified in disregarding them for the sake of simplicity. It may be noted, however, that, due to its (mostly) negative sign, this term may cause $S$ to become negative for very large values of $R_0 \sqrt{\omega/2D_c}$. At that point, the ensuing loss of stability (explained below) may be a real effect even if a quantitative prediction based on (2.19) may not be very accurate due to the assumption of spatial homogeneity in the bubble.

With this step, the expressions for $S$ and $D$ can be conveniently written as

$$S \simeq \frac{2b_l^2 R_c^2}{\rho_{0g} c_p T_w} F \left( \frac{\alpha R_c}{2D_c} \right)$$

and

$$D \simeq \frac{\mu_g^2 R_c^2}{\rho_{0g} c_p T_w} G \left( \frac{\alpha \rho_{0g}}{2D_c} \right),$$

where the functions $F$ and $G$ are given by

$$F(Z) = \begin{cases} \frac{Z^{3/2}}{(1 + \sqrt{Z})^2 + Z} & \text{if } Z < 1 \approx 0.5 \text{, has an error smaller than } 10\%, \text{ is} \\ \frac{1 + \sqrt{Z}}{(1 + \sqrt{Z})^2 + Z} & \text{if } Z > 1 \end{cases}$$

and

$$G(Z) = \begin{cases} \frac{Z}{3 + 2\sqrt{Z}} & \text{if } Z < 6.5 \text{, the function } G \text{ is monotonically decreasing and, for small and large values of the argument, respectively, it is approximated by} \\ \frac{1}{2\sqrt{Z}} & \text{if } Z > 6.5 \end{cases}$$

The function $F$ is monotonically increasing and, for small and large values of the argument, respectively, it is asymptotically approximated by

$$F(Z) \simeq Z^{3/2} \text{ and } F(Z) \rightarrow \frac{1}{2} Z^{1/2}. \quad (3.6)$$

A fit which, for $Z > 0.5$, has an error smaller than 10%, is

$$F(Z) \simeq \frac{Z}{3 + 2\sqrt{Z}} \quad (3.7)$$

and the error decreases monotonically past $Z = 6.5$. The function $G$ is monotonically decreasing and, for small and large values of the argument, respectively, it is asymptotically approximated by

$$G(Z) \approx 1 \text{ and } G(Z) \rightarrow \frac{1}{2\sqrt{Z}} \quad (3.8)$$

Graphs of these functions are provided in figure 1.

The factor multiplying the function $F$ in (3.3) has the dimensions of a pressure. In order to facilitate its comparison with the contribution $3k \rho_l g_0$ of the permanent gas in (3.1), we show in figure 2 a graph of this quantity normalized by the vapour saturation pressure $\rho_{sat}$:

$$P_v = \frac{2b_l^2 R_c^2}{\rho_{0g} c_p T_w} \frac{1}{\rho_{sat}},$$

as a function of temperature for water and liquid oxygen; this
is a number of the order of $10^{-2}$ or smaller over most of the temperature range shown.

In the linear approximation, stable oscillations about an equilibrium radius require a positive stiffness of the bubble contents, namely $\alpha_0 > 0$ or, from (3.1),

$$S > \frac{2\sigma}{R_0} - 3\kappa p_{g0}. \tag{3.10}$$

When this condition is not satisfied, the equilibrium condition $R = R_0$ is unstable.

The nonlinear phenomenon of rectified heat transfer is an important process which modifies these linear-theory predictions: a linearly stable bubble actually grows due to rectified heat transfer, the faster the higher the pressure amplitude, while the rate of growth or collapse of an unstable bubble would be modified with respect to the linear-theory prediction [39]. Nevertheless, (3.10) embodies the essence of the fundamental physical requirement of bubble stability.

Upon expressing $S$ as in (3.3), (3.10) leads to

$$F\left(\frac{\alpha_0 R_0^2}{2D_t}\right) \geq \frac{L}{R_0} \left(1 - \frac{3\kappa p_{g0} R_0}{2\sigma}\right), \tag{3.11}$$

where the characteristic length $L$ is defined by

$$L = \frac{\rho v_c T_w \sigma}{k_v \rho_c} \tag{3.12}$$

This quantity is shown by the solid lines as a function of temperature for water in figure 3 and for liquid oxygen in figure 4. For both fluids it is a very rapidly decreasing function of temperature.

For clarity we first consider the implications of the stability condition for bubbles containing a pure vapour, and then turn to bubbles with a mixture of vapour and permanent gas.

### 3.1. Pure vapour bubbles

With $p_{g0} = 0$, the stability condition (3.11) becomes

$$F\left(\frac{\alpha_0 R_0^2}{2D_t}\right) \geq \frac{L}{R_0}. \tag{3.13}$$

Equality of the two sides of this relation marks what has been called the ‘second resonance’ of vapour bubbles (e.g. [40]). Actually, this is not a true resonance, but simply the condition where the effective ‘stiffness’ of the bubble, proportional to its effective resonance frequency $\alpha_0^2$, vanishes [30]. In linear theory, it marks the transition from bubbles unable to execute stable linear oscillations to bubbles able to do so. As $F \to 0$ with its argument, this condition will be violated at any temperature provided the frequency is sufficiently small. While this conclusion is applicable to single bubbles, there actually is a lower limit to the frequency for which the present theory is applicable in the case of a bubbly liquid as discussed in §5.

At low temperature $L$ is very large and, therefore, the asymptotic approximation in the second one of (3.6) can be
used to find

\[ \frac{\omega R_0^2}{2\pi D_t} \geq \frac{4}{\pi} \frac{L^2}{R_0^2}. \]  

(3.14)

Conversely, at high temperature, \( L \) is small and, if the bubble radius is not too small, the asymptotic approximation in the first one of (3.6) is appropriate with the result

\[ \frac{\omega R_0^2}{2\pi D_t} \geq \frac{1}{\pi} \left( \frac{L}{R_0} \right)^{2/3}. \]  

(3.15)

For intermediate temperatures we can use the approximation (3.7) to invert this relation to find the approximate result

\[ \frac{\omega R_0^2}{2\pi D_t} \geq \frac{L}{\pi R_0} \left( 3 + \frac{2}{R_0} + \frac{2}{\sqrt{R_0^2 + 3}} \right). \]  

(3.16)

This relation reproduces (3.14) for large values of \( L/R_0 \), but not (3.15) as the starting point (3.7) is inaccurate for small values of the variable.

The stability boundary defined by (3.13) is shown in the form of frequency versus temperature for several values of the bubble radius in figure 5 for water and figure 6 for liquid oxygen. For any fixed frequency, there is a range of bubble radii, extending from 0 to some finite value, where the stability condition is not satisfied: when bubbles are too small, the latent heat absorbed and released in the course of the oscillations is insufficient to cause a pressure change capable of providing a positive stiffness. Conversely, for any given radius, stable oscillations are only possible for sufficiently large frequencies: when the oscillations are too slow, there is enough time for heat to be conducted away from or to the bubble surface, the vapour pressure hardly changes and no positive stiffness can be achieved. A striking feature of these curves is their extremely rapid rate of variation with temperature, which is essentially a consequence of the exponential temperature dependence of the saturation pressure.

The true resonance frequency of a pure vapour bubble is found from (3.1) by evaluating \( S \) at \( \omega_0 \). With the neglect of surface tension this is

\[ \omega_0^2 = \frac{2\rho_v \rho_c^2}{\rho_c^2 \chi_p T_m R_0^2} F \left( \frac{\omega_0 R_0}{2D_t} \right). \]  

(3.17)

Upon using the large-argument approximation for \( F \) we find

\[ \omega_0 \simeq \left( \frac{\rho_c^2 \chi_p T_m}{\rho_v R_0^2 D_t} \right)^{2/3} \frac{1}{R_0^{2/3}}. \]  

(3.18)

This relation was derived in Hao & Prosperetti [30] by an approximate argument which attributes the restoring force acting in the course of the oscillations of a vapour bubble to the pressure rise and fall associated with the release or absorption of latent heat. The proportionality to \( R_0^{-2/3} \) is in striking contrast to the well-known case of a pure gas bubble for which \( \omega_0 \propto R_0^{-1} \).

The ratio of the viscous to thermal damping is given by

\[ \frac{4\mu_k}{\rho_v R_0^2 D_t} = \frac{4\mu_k D_t \chi_p T_m}{\rho_v R_0^2 G(\omega R_0^2/2D_t) R_0} = \frac{B^2}{G(\omega R_0^2/2D_t) R_0^2}, \]  

where the characteristic length \( B \) is defined by

\[ B^2 = \frac{4\mu_k D_t \chi_p T_m}{\rho_v R_0^2}. \]  

(3.19)

A graph of \( B \) versus temperature is shown in figure 3 for water and in figure 4 for liquid oxygen. For small and large values of the argument of the function \( G \), (3.19) is, respectively,

\[ \frac{4\mu_k}{\rho_v R_0^2 D_t} \simeq \frac{B^2}{R_0^2} \]  

and

\[ \frac{4\mu_k}{\rho_v R_0^2 D_t} \simeq \frac{B^2}{2R_0} \sqrt{\frac{\omega_0}{2D_t}}. \]  

(3.20)

3.2. Gas–vapour bubbles

The stability condition (3.11) is always satisfied if the right-hand side is negative, i.e. if

\[ R_0 \geq \frac{2\sigma}{5k p_g} = \frac{2\sigma}{3\rho_{sat} \kappa p_g}. \]  

(3.22)

For water, the first factor \( 2\sigma/(3\rho_{sat}) \) falls from 85 \( \mu \)m at 0°C to 0.38 \( \mu \)m at 100°C while, for oxygen, it falls from 115 \( \mu \)m at 54 K to 2.52 nm at 125 K. At low temperatures the saturation vapour pressure \( p_{sat} \) is very small and it is therefore seen that even a relatively small amount of permanent gas can easily stabilize the bubble at all frequencies, in marked contrast with the case of pure vapour bubbles. This remark illustrates
the crucial importance of dissolved gases in the usual conditions under which hydrodynamic cavitation takes place.

When (3.22) is not satisfied, the stability condition cannot be met for all frequencies, but stability limits similar to those shown in figures 3 and 5 will exist. Qualitatively, it is seen in (3.11) that a permanent gas results in a decrease of the characteristic length \( L \) which, from figures 3 and 4, is equivalent to a temperature increase. It is then concluded from figures 5 and 6 that the gas contribution is stabilizing, as expected.

4. Bubbly liquids

With the results of the previous section we can now turn to the analysis of the phase speed and attenuation of pressure waves in a liquid containing gas–vapour bubbles.

Let \( k_m = \omega/c_m \) be the complex wavenumber of a pressure perturbation \( p' \) having a (real) angular frequency \( \omega \) and propagating in some direction \( x \). The space–time dependence of \( p' \) has the form

\[
p' \propto \exp(i(\omega t - k_m x)) = \exp(\text{Im} k_m x) \exp(i\omega \left(t - \frac{\text{Re} k_m}{\omega} x\right)),
\]

from which the (real) phase speed \( V \) and the rate of attenuation \( \alpha \) follow as

\[
V = -\frac{\omega}{\text{Re} k_m} \quad \text{and} \quad \alpha = -\text{Im} k_m.
\]

If we set

\[
c_r = \frac{k_m}{k} = u - iv,
\]

then

\[
V = \frac{c_r}{u} \quad \text{and} \quad \alpha = \frac{\omega}{c_r} v.
\]

The attenuation rate \( \alpha \) is related to the frequently used attenuation coefficient \( A \) by

\[
A = 20(\log_{10}e)\alpha \approx 8.68589 \frac{\alpha \omega}{c_r}.
\]

Upon separating the real and imaginary parts of the expression (1.2) for the complex phase speed, \( c_r^2/c_m^2 = C - iD \), with

\[
C = 1 + \frac{3\beta G}{R_0} \left( \frac{\omega^2}{\omega^2 - \omega^2} \right)^2 + 4b^2 \omega^2 \quad \text{and} \quad D = \frac{3\beta G}{R_0} \left( \frac{2b \omega}{\omega^2 - \omega^2} \right)^2 + 4b^2 \omega^2,
\]

it is readily found that

\[
u^2 = \frac{1}{2} \left( C + \sqrt{C^2 + D^2} \right) \quad \text{and} \quad v = \frac{D}{2u}.
\]

As from its expression (4.7), \( D > 0 \), it follows that \( v > 0 \) and \( \alpha > 0 \) as well, as expected.

4.1. Low-frequency behaviour

In most practical situations of flow cavitation the frequency of the pressure perturbation is low compared with the resonance frequency of the bubbles and it is therefore interesting to consider this limit explicitly.

As \( \omega \to 0 \) the gas tends towards an isothermal behaviour so that \( \kappa \simeq 1 \). Upon using the low-frequency approximations to \( F \) and \( G \) given in (3.6) and (3.8), we see that

\[
\omega^2 - 2b \omega + \frac{3g}{R_0} \frac{2g}{R_0} + \frac{2b^2}{c_p T_0} \left( \frac{\omega}{2D} \right)^{3/2} + \frac{4 \mu C_0}{R_0} e_i T_0 \right) \omega.
\]

When they do not vanish, the first two terms are the leading order contribution so that (1.2) reduces to

\[
c_r^2/c_m^2 = 1 + \frac{\beta \rho g_0}{\mu} \frac{1}{1 - 2\alpha/(3R_0 \rho_0)}.
\]

As \( \rho g_0 \sim 10^4 \), except for exceedingly small values of the bubble volume fraction, provided the bubbles are large enough that surface tension effects are negligible and \( \rho_0 \neq 0 \), we find

\[
V \approx \frac{\rho g_0}{\beta \rho_r} \left( \frac{2}{3} \right)^3 \frac{\sqrt{2}}{4}.
\]

In the conditions stipulated, with no vapour, \( \rho_0 = \rho_\infty \) and this expression reduces to the well-known result for the low-frequency phase speed in a liquid containing pure gas bubbles (e.g. [15,23]). The previous derivation shows that the result is valid also in the presence of vapour. This conclusion may explain the findings of the experiment of Shamsborhan et al. [12] whose measured speed of sound was in close agreement with (4.11) in spite of the presence of vapour, as mentioned in the Introduction. As the experiments were conducted at room temperature, \( \rho_\infty \) was very small and vapour had fairly small effects. Another remark concerning (4.11) is that, according to the equilibrium condition (2.11), \( \rho_0 \) equals \( \rho_\infty \). Thus, in the presence of vapour, \( \rho_0 \) is lower than the ambient value and the phase velocity will be even lower than the already very low values that it can achieve in the absence of vapour. This effect is the stronger the higher the temperature.

In the absence of gas, and disregarding surface tension effects, use of the expression (4.9) leads to

\[
\frac{c_r^2}{c_m^2} = 1 - \frac{3 \beta G}{2bR_0 \omega}.
\]

If viscosity is disregarded we then find

\[
V \approx \frac{h_v \rho_r R_0}{\rho_\infty} \left( \frac{2}{3} \right)^{1/2} \frac{2}{3} \left( \frac{2}{3} \right)^{1/2} \frac{\sqrt{2}}{4} \frac{\beta \omega c_p T_0}{h_v \rho_r R_0}.
\]

If thermal damping is also disregarded, it is the term proportional to \( \omega^{3/2} \) that dominates in (4.9) so that \( \alpha = 0 \) while

\[
V \approx \frac{2b^2 \omega^{3/2}}{3 \beta G h_v R_0^2} \left( \frac{\omega}{2D} \right)^{3/4}.
\]

In both these limits the phase speed becomes slower and slower with decreasing frequency as there is more and more time for the latent heat of condensation to be conducted away into the liquid so that the restoring force gradually decreases, as expected.
Figure 7. Low-frequency behaviour of the phase speed in water in the absence of gas, surface tension and damping for $R_0 = 200 \mu m$ and $\beta = 1\%$. In ascending order the curves are for $T = 20, 60, 80$ and $100^\circ C$; the ambient pressure is the saturation pressure at the corresponding temperature. (Online version in colour.)

disappears. Examples of this behaviour for water at $T = 20, 60, 80$ and $100^\circ C$, with $p_w = p_{sat}$ are shown in figure 7. Although these results are formally correct, they must be taken with some reservation for a subtle reason which is addressed in §5.

4.2. Near the stability limit

The low-frequency behaviour just discussed is considerably modified by surface tension effects. Let us consider in some detail the situation in the neighbourhood of the stability limit (3.11), where $\omega_0$ is very small. From (4.6) we find

$$ C \approx 1 - \frac{3\beta C_j^2}{R_0} \frac{1}{\omega^2 + 4\beta^2}, \quad (4.15) $$

which is typically large and negative so that $D/|C| \approx 2b/\omega$. A rather rough calculation based on the large-argument approximation for the function $G(Z)$ shows that $|C| > D$ below the true resonance frequency $\omega_0$, a conclusion that is confirmed numerically. To see what happens in these conditions, it is instructive to consider in (4.8) the limit $D \ll |C|$ when $C < 0$. We readily obtain

$$ u \approx \frac{D}{2|C|} \quad \text{and} \quad v \approx \sqrt{|C|}, \quad (4.16) $$

so that

$$ V \approx \frac{2\sqrt{|C|}c_i}{D} \quad \text{and} \quad \alpha \approx \frac{\alpha \sqrt{|C|}}{c_i}. \quad (4.17) $$

Upon substituting (4.15) and the analogous expression for $D$ we find

$$ V \approx \frac{\omega R_0}{b\sqrt{3\beta C_j^2 + 4\beta^2}} \approx \frac{R_0}{b\sqrt{3\beta}}, \quad (4.18) $$

$$ \alpha \approx \frac{\omega}{R_0} \sqrt{\frac{3\beta}{\omega^2 + 4\beta^2}} \approx \frac{\sqrt{3\beta}}{R_0}. $$

Remarkably, the wave propagation rests mainly on the damping of the bubble motion. This conclusion can be understood by noting that, as the restoring force effectively vanishes, with no damping the bubble motion would lag the excitation by $\pi$, which would make the medium incompressible as signalled in (4.18) by the divergence of the phase velocity. It is only when the presence of damping reduces the phase lag and restores some compressibility that waves can propagate.

With the parameter values used in figure 8 for water, $\omega_0^2$ is negative all the way up to about $289.3 \, kHz$ for $T_w = 20^\circ C$ while, for $T_w = 60$ and $80^\circ C$, it becomes positive at frequencies of $137.6 \, Hz$ and $9.92 \, Hz$. As for $C$, for $T_w = 20^\circ C$ it remains negative all the way to the high-frequency limit (4.20) discussed below, while, for $T_w = 60$ and $80^\circ C$, it becomes positive at $151.5 \, Hz$ and $9.93 \, Hz$, respectively. For the liquid oxygen example of figure 9, the bubble stiffness becomes positive at frequencies of $37.39 \, kHz$ and $9.92 \, kHz$ while the thermal damping vanishes.

4.3. High-frequency behaviour

Since, as follows from (3.1), (3.3), (3.4) and (3.6), for high frequencies $\omega_0^2 \ll \omega$ while the thermal damping vanishes.
proportionally to $1/\sqrt{\omega}$, acoustic damping constitutes the leading contribution to $b$ and, for large $\omega$, we have

$$C \approx 1 - \frac{3\beta^2}{R_0^2\omega^2 (R_0^2/c_s^2) \omega^2 + 1},$$

(4.19)

which is negative for $\omega$ lower than

$$\omega \approx \left(\frac{\sqrt{1 + 12\beta} - 1}{2}\right)^{1/2} c_s \frac{R_0}{R_0^2}.$$  (4.20)

Past this point $C \rightarrow 1$ from below and so does $u$, with the consequence that $V \rightarrow c_s$ from above. Aside from the weak dependence of $c_s$ on temperature, this behaviour is independent of the liquid temperature. Bubbles only enter through their volume fraction and their composition is irrelevant.

With the parameter values used in figure 8 for water, (4.20) corresponds to a frequency $\omega/2\pi \approx 203.8$ kHz while, with the values of figure 9 for liquid oxygen, it corresponds to about 143.5 kHz.

**4.4. Numerical results—phase speed**

Examples of the dependence of the phase speed of pressure waves on frequency in the absence of a permanent gas are shown for water at $T = 20$, 60, 80 and 100°C in figure 8 and for liquid oxygen at $T = 60$, 80 and 90 K in figure 9; in both cases the bubble radius is $R_0 = 200 \ \mu$m and the volume fraction $\beta = 1\%$; the ambient pressure equals $p_{sat}(T) - 2\sigma/R_0$. The horizontal dashed line is the speed of sound in the liquid without bubbles at $T = 20°C$ and 60 K, respectively; to avoid encumbering the graphs with too many lines we have taken $c_s$ independent of temperature. The portion of the curves shown by the dashed lines correspond to the range where the effective resonance frequency $\omega_0/C$ is negative.

The key to understanding the behaviour of these results is to focus on the two frequency regions where the quantity $C$ defined in (4.6) can become negative. The first one is at low frequencies near the stability limit, as discussed earlier in §4.2, where the effective resonance frequency vanishes. The second one is at high frequency, where $\omega^2 > \omega_0^2 > 0$ as discussed in §4.3. The situation is clearer if the parameter $D$ is assumed to be small. In this case, as noted before in connection with (4.16), $u$ can become smaller than 1 and the phase speed $V$ will then exceed the pure-liquid value $c_s$.

The descending portion of the curves for $T = 20$, 60 and 80°C of figure 8 and for $T = 60$ K in figure 9 show the gradual decrease from the peak corresponding to the stability limit associated with the first region where $C < 0$. This behaviour occurs at lower frequencies for the higher-temperature cases and is not visible in the frequency range shown. As the true resonance frequency is approached, the phase speed decreases as the bubble oscillations tend to become in quadrature with the driving. Past the resonance frequency the phase increases above $\pi/2$, the bubbly medium becomes less and less compressible, $C$ becomes negative again and once more the phase speed exceeds the pure-liquid value, which is however recovered as $C$ approaches 1 from below as shown in §4.3. For the case of gas bubbles, this latter behaviour has been confirmed in the experiments of Cheyne et al. [24].

The addition of a permanent gas changes drastically the dependence of the phase speed upon frequency as can be seen in figures 10 and 11, both for water. In the first one the gas partial pressure equals the vapour saturation pressure while, in the second one, the gas partial pressure equals 20%, 50% and 100% of the vapour saturation pressure; all other conditions and parameter values are the same as in figure 8. The strong effect of the permanent gas is particularly striking as can be seen by comparing the three curves of figure 11.

The frequency range where $C < 0$ at low frequencies disappears and the phase speed is very close to the value given by (4.11) until there is a pronounced dip in correspondence of the resonance frequency of the bubbles. At still higher frequencies the phase speed exceeds the speed of sound in the pure liquid as before.

**4.5. Numerical results—attenuation**

The attenuation parameter $\alpha$ in the absence of a permanent gas is shown for the same cases of figures 8 and 9 in figure 12 for water and figure 13 for liquid oxygen. The curves exhibit a broad high plateau at levels of the order of 1000 m$^{-1}$ at intermediate frequencies. To the left of the plateau, the attenuation strongly increases with decreasing...
temperature, while to the right temperature effects are minor similarly to what is observed for the phase speed. The growth before the broad plateau is mostly due to the thermal component of the damping, while the acoustic component dominates from about the middle of the plateau on. If acoustic damping is removed, the maximum level reached changes little, but the decline at high frequencies is much steeper. The effect of viscous damping is mostly small.

The addition of a permanent gas has again a very strong effect on these results as shown in figure 14 for water under the same conditions used in figure 10. Below the plateau, attenuation decreases by several orders of magnitude compared with the pure vapour case because the restoring force provided by the gas limits the amount of evaporation and condensation and the associated thermal dissipation. The smaller thermal damping provides a sharper and narrower resonance and the corresponding maximum attenuation is larger than without gas, even though this may be an artefact of the model as observed earlier in §1. The level of the plateau is comparable to the one found without gas.

It may be noted that these results do not include the contribution of the thermal damping due to the gas and, therefore, are likely to underestimate somewhat the actual attenuation.

5. Discussion of the low-frequency limit (4.14)

In §64 of Landau & Lifshitz [41] the authors derive the following approximate expression for the speed of sound in a liquid containing vapour bubbles:

\[
V_{LL} = \frac{h \nu \rho_v}{\rho_l \sqrt{c_p \rho_l T_w}}.
\]  

This result, which is shown as a function of temperature for water in figure 15 and liquid oxygen in figure 16, is obtained in the limit of small vapour volume fraction, assuming thermodynamic equilibrium, that the vapour behaves like an ideal gas, that \( \rho_v \ll \rho_l \) and ignoring surface tension effects. It should therefore be comparable with our result (4.14) although, superficially, it looks quite different as it does not depend on the vapour volume fraction or on frequency. Upon comparing the two expressions we see that they
would agree for \( \omega \) equal to the angular frequency

\[
\omega_{\infty} = 2D_\infty \left( \frac{3B^2}{2K_0} \right)^{2/3} = 18^{1/3}D_\infty \left( \frac{B^{1/3}}{K_0} \right)^2.
\]

(5.2)

As the mean inter-bubble distance is of the order of \( R_0/\sqrt{\omega_{\infty}} \), it is clear that \( 1/\omega_{\infty} \) is of the order of the diffusion time of heat over a distance comparable to the mean inter-bubble distance. This remark makes it easier to understand a limitation inherent in our low-frequency result (4.14) which may not have been apparent at first sight. In the derivation given in §2.2 it was assumed that, far from the bubble, the liquid temperature had the undisturbed value \( T_\infty \). Evidently, this assumption is reasonable as long as the diffusion length is smaller than the mean inter-bubble distance, but it becomes invalid when this condition is violated. Thus, our result (4.14) becomes invalid for frequencies lower than \( \omega_{\infty} \) given in (5.2), past which it must be replaced by the frequency-independent expression (5.1).

This point becomes clearer by providing an alternative derivation of (5.1) different from that given in Landau & Lifshitz [41] but, of course, equivalent to it.

Consider a certain volume \( V = V_I + V_v \), occupied in part by liquid with a small amount of vapour, \( V_v \ll V_v \), and assume that a perturbation causes the condensation of a small amount of vapour \( dm_v \). In adiabatic conditions, the first principle of thermodynamics requires that

\[
h_{\infty} dm_v + d[(c_p\rho_I V_I + c_p\rho_v V_v)T_\infty] = V dp.
\]

(5.3)

As \( dm_v = d(\rho_I V_I) = -d(V_v \rho_v) \), \( \rho_v V_v \ll \rho_I V_I \), use of the Clausius–Clapeyron relation with \( \rho_v \ll \rho_I \) leads to

\[
h_{\infty} \left( 1 + \frac{c_p - c_{pv}}{h_{\infty}} T_\infty \right) dm_v = -\frac{\rho_v V_v c_{pv}T_\infty}{h_{\infty} \rho_v} \left( 1 - \frac{\rho_v h_{\infty}}{\rho_I c_{pv} T_\infty} \right) V dp.
\]

(5.4)

We now approximate this relation further by dropping the second term in both parentheses as, far from the critical point, \( (c_p - c_{pv}) T_\infty \ll h_{\infty} \) and \( \rho_v h_{\infty}/(\rho_I c_{pv} T_\infty) \ll 1 \) given the smallness of \( \rho_v/\rho_I \); the result is

\[
\frac{dm_v}{dp} \simeq -\frac{\rho_v V_v c_{pv}T_\infty}{h_{\infty}^2 \rho_v}.
\]

(5.5)

We are interested in

\[
\frac{1}{c^2_m} = \frac{d^2p}{dp^2} = -\frac{\rho_m}{V} \frac{dV}{dp} \simeq -\frac{\rho_v}{V} \frac{dV_v}{dp}.
\]

(5.6)

From the equation of state of perfect gases, again using the Clausius–Clapeyron equation, we have

\[
\frac{dV_v}{dp} = \frac{V_v}{m_v} \frac{dm_v}{dp} + \frac{V_v}{h_{\infty} \rho_v} \frac{V_v}{p},
\]

(5.7)

or, with the previous expression for \( dm_v/dp \),

\[
\frac{dV_v}{dp} \simeq -\frac{\rho_v V_v c_{pv}T_\infty}{h_{\infty}^2 \rho_v} + \frac{V_v}{h_{\infty} \rho_v} \left( \frac{p}{h_{\infty} \rho_v - 1} \right).
\]

(5.8)

As \( V_v \) is small, the second term can be dropped and, upon substituting into (5.6), we find

\[
\frac{1}{c^2_m} \simeq \frac{\rho_v^2 c_{pv} T_\infty}{h_{\infty}^2 \rho_v},
\]

(5.9)

which is (5.1). The approximations used in this derivation are similar to those on which (5.1) rests.

It will be appreciated that the key difference between the physical model adopted in this derivation and the one of §2.2 is that, here, the temperature of the entire liquid mass has been assumed to change as a consequence of the latent heat associated with the change of the vapour mass, rather than only a shell surrounding each bubble as in (2.15).

Competing interests. I declare I have no competing interests.

Funding. This study has supported by the US National Science Foundation under grant no. CBET 1335965.

Acknowledgement. I am grateful to Prof. Chris Brennen for some useful suggestions.

References


