Triply periodic minimal and constant mean curvature surfaces

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We want to summarize some established results on periodic surfaces which are minimal or have constant mean curvature, along with some recent results. We will do this from a mathematical point of view with a general readership in mind.

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1. INTRODUCTORY MATERIAL

1.1. Curvatures

Let us start with the definition of the curvature of a planar curve \( \Gamma \), oriented by its normal \( n \). Up to sign, its curvature \( \kappa \) at a given point is the inverse of the radius \( R > 0 \) of a circle which agrees with the curve at the point up to second order; if the curve coincides to second order with a straight line, then we define \( \kappa = 0 \). The curvature is positive provided the circle has its centre to the side of the normal \( n \), and negative otherwise.

The standard definition of curvature for a surface uses the notion of curvature for curves as follows. At any given point \( p \) of a surface \( \Sigma \) contained in Euclidean space \( \mathbb{R}^3 \), the surface normal \( n \) and any tangent direction \( v \) span a normal plane. The surface intersects the normal plane in a curve \( \Gamma \), which is oriented by the surface normal. The curvature of \( \Gamma \) at \( p \) is called the normal curvature of the surface in direction of the tangent direction \( v \). At any point \( p \in \Sigma \), the normal curvature attains a maximal and a minimal value \( \kappa_1(p), \kappa_2(p) \), called principal curvatures for two orthogonal tangent directions: \( \kappa_1(p) \perp \kappa_2(p) \). In order to understand orthogonality, we write the surface as a graph \((x, y, f(x, y)) \) over its tangent plane, and then use the Taylor expansion for \( f \). Its second-order term involves the symmetric Hessian matrix \( (\partial^2 f) \), whose minimal and maximal eigenvalues are the principal curvatures, and they are indeed attained at orthogonal eigendirections. The average of the principal curvatures is the mean curvature,

\[
H := \frac{1}{2}(\kappa_1 + \kappa_2),
\]

moreover \( K := \kappa_1 \kappa_2 \) is the Gauss curvature. If \( H(p) = 0 \), then the \( \kappa_i \) have opposite signs, and so \( p \) is a saddle point of the surface such that \( K(p) \leq 0 \).

1.2. Parallel surfaces

We present another way to introduce \( H \), which is relevant to our purpose. Let us start with the case of a curve \( s \mapsto \Gamma(s) \). Let \( \Gamma'(s) := \Gamma(s) + tn(s) \) be the parallel or offset curve at (constant) distance \( t \) from \( \Gamma = \Gamma^0 \). For example, if \( \Gamma \) is a circle of radius \( R \) with inner normal \( n \), then \( \Gamma' \) is the circle of radius \( R - t \), meaning that the length element scales with \( (R - t)/R \) when we go from \( \Gamma \) to \( \Gamma' \). If a general curve \( \Gamma \) is approximated by a circle of radius \( R = 1/\kappa \), then its parallel curve is approximated by a circle of radius \( R - t \). Therefore, also for a general curve \( \Gamma \), the length elements relate as

\[
ds^2 := \frac{R - t}{R} ds = \left(1 - \frac{t}{R}\right) ds = (1 - t\kappa) ds. \tag{1.1}
\]

We view this equation as to say that curvature is the first-order change in length when going from a curve to its parallel curves. In fact, (1.1) says it is the negative of this number, and indeed the parallel curves to a positively curved curve become shorter. Upon integration, we obtain

\[
\text{length}(\Gamma') = \text{length}(\Gamma) - t \int \kappa(s) ds.
\]

We apply the same approach to the area element of a parallel surface

\[
\Sigma'(u, v) := \Sigma(u, v) + \kappa t(u, v).
\]

We need two assertions: first, the length elements \( ds_1 \) and \( ds_2 \) in the two orthogonal principal curvature directions multiply to the area element \( dS = ds_1 ds_2 \). Second, the parallel surface \( \Sigma' \) has the same normals and the same normal planes, which it intersects in parallel curves. In particular, the length elements \( ds_1^t \) and \( ds_2^t \) of the parallel surface can be computed by (1.1), and they are again orthogonal. This gives

\[
ds^t = ds_1^t \cdot ds_2^t = ds_1(1 - t\kappa_1) \cdot ds_2(1 - t\kappa_2),
\]

\[
eq ds_1 ds_2(1 - t(\kappa_1 + \kappa_2) + t^2\kappa_1\kappa_2)
\]

\[
eq dS(1 - t^2 H + t^4 K). \tag{1.2}
\]

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and so mean curvature is, up to a factor, the first-order change of the area element. Again, we can integrate and obtain this time an expression for area,

$$A(\Sigma^1) = A(\Sigma) - 2t \int_\Sigma H dS + t^2 \int_\Sigma K dS. \quad (1.3)$$

### 1.3. Periodic surfaces

For the present study, a surface is connected and has no boundary. Here, we will consider oriented surfaces only. In general, we admit self-intersections, but for most cases our surfaces are embedded instead. In this case, they are necessarily oriented.

In \( \mathbb{R}^3 \), a lattice of rank \( k \) is a set \( \Lambda := \{ a_1 v_1 + \cdots + a_k v_k : a_i \in \mathbb{Z} \} \), where the vectors \( v_1, \ldots, v_k \) are linearly independent. Therefore, \( k \leq n \) and so for \( \mathbb{R}^3 \) only \( k = 1, 2, 3 \) can arise.

**Definition.** A surface \( \Sigma_0 \subset \mathbb{R}^3 \) is singly, doubly or triply periodic if it is invariant under translations of some lattice \( \Lambda \) with rank 1, 2 or 3, respectively. For the present study, we require that the quotient surface \( \Sigma \) be compact.

Our abstract quotient surface \( \Sigma_0/\Lambda \) is contained in the space quotient \( \mathbb{R}^3/\Lambda \). The pair can be represented in a non-unique way with a concrete fundamental domain contained in a fundamental cell. For instance, \( \mathbb{R}^3/\mathbb{Z}^3 \) is a 3-torus, which can be represented by a cube whose opposite faces are considered to be identified; many figures of the present study represent this case. By the identification, neither the fundamental cell nor the fundamental domain has boundary. Note also that we do not require that \( \Lambda \) contains all translational symmetries, so that the quotient surface may be chosen larger than necessary. For example, the cylinder is simply periodic with a lattice generated by any non-zero axis-parallel vector, \( v \).

**Compactness** is not standard to assume in the above definition, but will be essential for us. We do not want to give a formal definition here, but let us say that it implies finite area of the quotient surface. For reasonable examples, it is actually equivalent to it.

A mathematical fundamental cell \( \mathbb{R}^3/\Lambda \) may be chosen smaller than the crystallographic unit cell because we require that it only generates the entire space by translations. Examples are the body centred cubic (BCC) and face centred cubic (FCC) lattices.

### 1.4. Genus

The genus measures the topological complexity of a surface.

**Definition 1.1.** The genus \( g \) of an orientable surface \( \Sigma \) or, in the periodic case, of the quotient surface \( \Sigma = \Sigma_0/\Lambda \) is the number of handles needed to attach to a sphere, up to deformation.

For the case of a periodic surface, the genus is always finite. For instance, the \( P \) surface is regarded as a sphere with a handle in each of the three coordinate directions, so it has genus 3 (figure 1).

The Gauss–Bonnet formula relates the total curvature to the genus

$$\int_\Sigma K dS = 2\pi(2 - 2g). \quad (1.4)$$

We will apply it to quotient surfaces \( \Sigma = \Sigma_0/\Lambda \).

### 1.5. Minimal and constant mean curvature surfaces

By definition, a surface is minimal if its mean curvature vanishes, \( H = 0 \). Equivalent is

- it is critical for area, or
- small pieces have minimal area when compared with pieces with the same boundary.

By (1.3), the first property certainly holds in the parallel surface family because \( d/dt A(\Sigma^1)_{t=0} = -2 \int_\Sigma H = 0 \). A more general formula shows that it also holds for arbitrary deformation families.

By definition, a surface with \( H \) constant is a constant mean curvature (CMC) surface. Equivalent is

- it is critical for area for all variations which preserve the enclosed volume, or
- small pieces have minimal area when compared with pieces which enclose the same volume and have the same boundary.

## 2. TRIPLY PERIODIC MINIMAL SURFACES

### 2.1. A remark on the simply and doubly periodic case

Our compactness assumption rules out any simply periodic minimal surfaces and implies that doubly periodic minimal surfaces must be planar. Were we to allow for non-compact fundamental domains, we had many such examples, for instance Scherk’s singly and doubly periodic minimal surfaces.

To explain this fact, we need the maximum principle, which prohibits one-sided touchings of minimal surfaces:

**Theorem 2.1.** A minimal surface cannot have a point of tangency with another minimal surface which stays to one side of the first surface, unless the two surfaces agree.
Now, consider a doubly periodic surface, $\Sigma_k$. Its lattice $\Lambda = \{a_1v_1 + a_2v_2 : a_i \in \mathbb{Z}\}$ defines a normal vector $N \perp (v_1, v_2)$. We claim that the height function $\langle \cdot, N \rangle$ in direction $N$ is bounded on $\Sigma_k$. To understand this, represent the fundamental cell $\mathbb{R}^3/\Lambda$ by a generalized cylinder in direction $N$ over the planar parallelogram with vertices $0, v_1, v_2, v_1 + v_2$. It follows from our compactness assumption that the height $\langle \cdot, N \rangle$ is bounded on the fundamental domain $\Sigma_k/\Lambda$, but by periodicity this proves the claim. Now, we claim there is a plane $P \subset \mathbb{R}^3$, perpendicular to $N$, which touches $\Sigma_0$ from one side. Indeed, our first claim implies that there is a plane (with large height) disjoint to $\Sigma_0$. Sliding it inwards towards $\Sigma_0$, we arrive at a particular plane $P$ which has a one-sided touching with $\Sigma_0$. Then, the maximum principle implies that $P$ must coincide with $\Sigma_0$.

Similarly, a singly periodic minimal surface must coincide with planes whose normal we are free to choose from a circle. Hence, no such surface can exist.

### 2.2. Examples

It is a fact, usually proven using the Weierstrass representation, that triply periodic minimal surfaces must have genus at least 3.

The first embedded examples are due to Schwarz in 1890 and his student Neovius in 1883. They used the Weierstrass representation formula to determine five triply periodic minimal surfaces, namely $P$, $D$, $H$, $CLP$ and $N$. Almost a century later, in 1970, Schoen [1] described the next surfaces: he exhibited 12 new surfaces, among them the gyroid. Rigorous proofs along with further surfaces were given by Karcher [2] (and for the embeddedness of the gyroid by Grosse-Brauckmann & Wohlgemuth [3]).

Since the 1980s, many more periodic surfaces have been found and described. For instance, starting in 1988, Fischer and Koch suggested many beautiful examples, obtained by completely classifying the systems of straight lines which can be contained in minimal surfaces. Mathematical existence proofs for the containing minimal surfaces involve work specific to the surface in question and have been carried out for some of these surfaces.

### 2.3. Classification problem

With many triply periodic minimal surfaces known by now, it becomes a task to classify them, at least for the case of a large symmetry group. In the following, we continue to assume embeddedness.

We expect that the number of embedded triply periodic surfaces with a given space symmetry group is always infinite. Indeed, there are series of computed surfaces with increasing genus, obtained by adding handles (figure 2). Also, any given triply periodic minimal surface, considered with a choice of increasing fundamental domains leads to the same observation.

So perhaps symmetry group together with genus could classify embedded triply periodic minimal surfaces. However, as the two different surfaces with genus 12 in figure 3 indicate, this is not the case. Nevertheless, the figure can suggest the following combinatorial classification of candidates for minimal surfaces. If we add a $k$-fold handle, then the site index of the location of the handle tells us how often we have to add this in order to be compatible with the symmetry group. For instance, the Neovius surface has fourfold handles, each contributing 3 to the genus, attached at the three-edge midpoints of the cubical unit cell; this gives $3 \times 3 = 9$ for the genus (cf. figure 1). If we list all combinatorial possibilities, taking into account which kinds of handles can be added once or several times, then we obtain a list of candidates for minimal surfaces for which existence (and perhaps uniqueness?) could be checked. Figure 3 shows the first eight such surfaces, ordered by their genus, for the group $Pm3m$.

It is a non-trivial problem to assert that a minimal surface is new and does not agree with known examples. The suggested classification by symmetry group, genus, handle type and site index of the attachment point gives a systematic way to decide this question.

Interesting problems about periodic minimal surfaces involve deformation issues. For a fixed lattice, a minimal surface usually appears rigid up to translation (at least I am not aware of any counterexample). Rigidity becomes a mathematical fact provided a condition is met: the only $A$-periodic solutions $Lu = 0$ of the Jacobi operator $L := \Delta u - 2K$ are those which are induced by translation. That is, on the quotient surface, the eigenspace of the eigenvalue 0 of the operator $L$ is required to be at most three-dimensional. The Jacobi operator arises from the second variation of the surface, where the surface is varied in the normal direction. However, when we allow for deformations of the lattice then surfaces such as $P$, $D$, $G$ can be deformed into one-another via minimal surfaces (results by Fogden-Hyde, Weber).

### 3. Naturally occurring periodic surfaces

Minimization of area caused by surface tension appears as a good reason for triply periodic minimal surfaces to occur in nature. However, this point of view is mistaken. First, by scaling the lattice larger, the area density...
Thus, the tails are arranged on a Gauss curvature of the surface. The result of area minimization alone.

But even after fixing the lattice, a minimal surface $S = S_0/\Lambda \subset \mathbb{R}^3/\Lambda$ is not a minimum of area.

**Theorem 3.1.** A triply periodic non-planar minimal surface $S$ is a strict maximum of area in its parallel surface family $S^\iota$.

**Proof.** Because $H = 0$, this follows from (1.3).

$$A(S^\iota) - A(S) = -2t \int \limits_{S^\iota} HdS + t^2 \int \limits_{S^\iota} KdS = 0 + t^2 2\pi(2 - 2g) < 0 \quad \text{for} \quad t \neq 0.$$

At the second equality sign, we have invoked the Gauss–Bonnet formula (1.4). The inequality follows from the fact that the genus is at least 3 for a non-planar periodic minimal surface.

Therefore, minimal surfaces can only form due to other reasons. In lipid systems, molecules self-arrange to layers forming periodic minimal surfaces. Here, the so-called spontaneous mean curvature is caused by molecules whose tail has a geometry different from its head. Assume that the molecule heads form a surface $S$, and their tails form the parallel surface $S^\iota$. Recall from (1.2) that $dS^\iota/dS = 1 - i2H + t^2K$. For a wedge-shaped molecule of small length $t$ whose cross-sectional area at the tail is $1 + c$ times larger compared with the head, first-order considerations give $c = -2tH$. Thus, the tails are arranged on a surface with CMC $H = -c/(2t)$; only for $c = 0$ do we have minimal surfaces, whereas in general CMC surfaces will model the interface.

It can be necessary to modify these ideas further. If the cross section of the molecules is different at head and tail, for instance, square or circular at the head but distinctly rectangular or elliptic at the tail, then principal curvatures with a certain ratio will be favoured. Taking this into consideration, (1.2) can be solved up to second order and will give a preferred Gauss curvature of the surface.

In addition, large principal curvatures which occur in minimal or CMC surfaces imply significant distortions of the molecular geometry and so may be unreasonable for actual interfaces. This lets us expect that the bending of the surface, represented at each point by $\kappa_1^2 + \kappa_2^2$, should be small.

We conclude that it is often appropriate to minimize energies involving geometric quantities and will come back to this problem in §5.

Beyond these effects are packing constraints. They become important whenever molecules are attached to an interface and fill some or all labyrinths. This is the case for the intermaterial dividing surfaces of diblock copolymers as well as for bilayers with molecules which have an amphiphobic side.

To return from this excursion of ideas leading to more involved energies, let us mention a much more trivial idea. Often the exact energy to be minimized is unknown, but it may still be desirable to have rather smooth embedded surfaces at hand, which attain a given genus, space symmetry group and given volume fraction of the two components (labyrinths) to the two sides of an surface. Here, smooth could be quantified in terms of weak higher Fourier modes. While minimal surfaces cannot satisfy a condition on the volume fraction, the families considered in §§4 and 5 can serve this purpose.

**4. CONSTANT MEAN CURVATURE FAMILIES OF PERIODIC SURFACES**

For a fixed lattice and space symmetry group, we will discuss families of CMC surfaces, where the values of $H$ attained can include the minimal case, $H = 0$. On the other hand, singly and doubly periodic CMC surface families cannot contain a minimal surface, as follows from the result mentioned at the beginning of §2.1.

Delaunay in 1841 determined singly periodic CMC surfaces as surfaces of revolution. One family is embedded, the so-called unduloids (figure 4). Owing
to work of Korevaar et al. [5], we know that no further singly periodic embedded CMC surfaces exist (again, the compactness assumption is crucial for this to hold). Lawson [6] proved the existence of two doubly periodic CMC surfaces. The author proved they are contained in a one-parameter-family of surfaces, and established a third such family [7] (figure 5). While computer calculations indicate these surface families are embedded, a mathematical proof has not been given up-to-date.

Triply periodic CMC surfaces have attracted particular attention because they can often be viewed as deformations of minimal surfaces. Anderson computed five such families numerically in 1986, namely \( P \), \( D \), \( N \), \( FRD \) and \( IWP \) [8]. For these symmetric surfaces with many mirror planes, the conjugate surface construction, developed by Karcher [2] and Grosse-Brauckmann [7], provides a strategy to prove existence. It confirms the previous numerical results (figures 6 and 7).

**Theorem 4.1.** There is a one-parameter family of \( Pm3m \)-symmetric CMC surfaces, including the minimal \( P \)-surface and degenerating in spheres. This family does not bifurcate to other CMC surfaces with the same symmetry [9].

It should be straightforward to apply the same ideas to the \( D \)-surface and other triply periodic surfaces. There is also work by Kapouleas [10] which proves the existence of periodic CMC surfaces without any need to assume mirror symmetries. However, his method gives surfaces which are close to a degenerate limit, in many cases touching spheres (as in the first and large images of figures 6 and 7). These surfaces have very small handles and their large principal curvature deviations make them seem physically less significant than the part of the family closer to the minimal surface.

The gyroid is perhaps the most famous triply periodic minimal surface. The gyroid symmetry group \( I \bar{4} 32 \) has no mirrors and therefore the previously mentioned numerical and theoretical methods fail for this surface. The discovery of this morphology in diblock copolymers in the 1990s led the author to study gyroids theoretically, and numerically using Brakke’s surface evolver, with the following results.

**Theorem 4.2.**

(i) Schoen’s minimal gyroid is embedded,

(ii) there exist gyroids with \( |H| \) nonzero small and
the entire family was computed with the Surface Evolver; it includes the minimal gyroid, and degenerates in touching spheres (figure 8) [3,11].

The data obtained show that CMC gyroids attain a large volume fraction ranging from 5.6 per cent to 94.4 per cent, thereby covering experimentally determined data for the polymer problem. For further pictures and numeric data, we refer to Grosse-Brauckmann [11].

Let us conclude this section by mentioning some problems.

— Does any triply periodic minimal surface give rise to a one-parameter family of surfaces with CMC? The answer is in the affirmative for the non-degenerate minimal surfaces which are defined as those where the operator \( \Delta \Sigma + \kappa_1^2 + \kappa_2^2 \) does have only a three-dimensional eigenspace for the eigenvalue 0 (as discussed in §3).

— How do these families end? Many symmetric examples degenerate in touching spheres, but this is not necessarily so.

— Do bifurcations occur? Again, this cannot happen at a non-degenerate CMC surface.

— Which portion of the family is stable for area under a volume constraint? This is known to be true for the minimal \( P, D \) and \( G \) surfaces, but cannot hold for their entire families.

5. HELFRICH, WILLMORE AND BENDING ENERGY FOR PERIODIC SURFACES

For \( a, b \geq 0 \) and \( c \in \mathbb{R} \), the Helfrich energy of a surface \( \Sigma \) is

\[
E(\Sigma) := \int_{\Sigma} a + bH^2 + cKdS = aA(\Sigma) + \int_{\Sigma} bH^2 + cKdS.
\]

This is the most general energy that can be written as a quadratic symmetric polynomial of the principal curvatures.

**Theorem 5.1.** The following energies have the same critical points (or minima) over surfaces of fixed genus \( g \):

— Willmore energy, \( W(\Sigma) := \int_{\Sigma} H^2dS \),

— bending energy, \( B(\Sigma) := \int_{\Sigma} \kappa_1^2 + \kappa_2^2dS \) and

— Helfrich energy, \( E(\Sigma) \) with \( a = 0 \) (no area term).

**Proof.** By the Gauss–Bonnet theorem (1.4),

\[
4W(\Sigma) = \int_{\Sigma} \kappa_1^2 + \kappa_2^2dS + 2\int_{\Sigma} KdS = B(\Sigma) + 4\pi(2 - 2g),
\]

so the difference of the energies is a constant. The same reason shows that the Helfrich minimizers are independent of the value for \( b \) and \( c \) chosen. 

(reproduced with permission from N. Bechtloff [12].)
The Willmore or bending energy is independent of scalings of the lattice (i.e. the lattice constant). This is a consequence of the fact that area scales by the square of the scaling factor, and $H^2$ or $k^2$ by one over the square. The Euler–Lagrange equation for Helfrich is the partial differential equation

$$2b(\Delta_2 H + 2H(H^2 - K)) - 4aH = 0.$$  

The term $\Delta_2 H$ has fourth order, which makes the mathematics of this equation hard. 

A triply periodic minimal surface $S$ naturally is an absolute minimizer for the Willmore or bending energy among surfaces with the same topology (genus), with $W(S) = 0$ or $B(S) = 4\pi(2g - 2)$. In fact, it is also critical for Helfrich, because $A(S)$ is critical.

Hence, we can study deformations of minimal surfaces by minimizing the Willmore or bending energy under a volume constraint, a situation analogous to the mean curvature problem. Moreover, different choices for the quotient $b/a$ in the Helfrich energy interpolate between Willmore or bending energy ($b/a = \infty$) and area functional ($b/a = 0$).

In Surface Evolver computations, the following has been verified.

**Theorem 5.2.** The Lawson, IWP, N and Karcher–Polthier surface of genus 12 are contained in a one-parameter family of surfaces minimizing Willmore or bending energy. The triply periodic surfaces exist for any volume fraction in $(0, 1)$ and degenerate with cylindrical shapes in a graph which coincides with Schoen’s skeletal graph, when defined [12].

Figures 9 and 10 show the $P$ and $G$ families where the Schoen skeletal graphs are the degenerate limits. In case of the Karcher surface of genus 12 (see fifth image of figure 3), a skeletal graph is not (uniquely) defined. So, the limit of the Willmore surfaces selects a skeletal graph. This problem needs further investigation.

We also remark that close to the minimal surface, the Willmore and CMC family are very similar, while at the degeneration they are substantially different.

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