The process of cutting is analysed in fracture mechanics terms with a view to quantifying the various parameters involved. The model used is that of orthogonal cutting with a wedge removing a layer of material or chip. The behaviour of the chip is governed by its thickness and for large radii of curvature the chip is elastic and smooth cutting occurs. For smaller thicknesses, there is a transition, first to plastic bending and then to plastic shear for small thicknesses and smooth chips are formed. The governing parameters are tool geometry, which is principally the wedge angle, and the material properties of elastic modulus, yield stress and fracture toughness. Friction can also be important. It is demonstrated that the cutting process may be quantified via these parameters, which could be useful in the study of cutting in biology.

1. Introduction

Cutting may be defined as a process in which new surfaces are created in a solid by the use of a tool. Such tools are usually sharp though, in reality, blunt tools can cut and the sharpness can be an important factor in the process. The creation of new surfaces in a solid has important connotations in mechanics because it implies a change in continuity such that there will be an energy change. This notion was first explored by Griffith [1] and he introduced the parameter energy release rate ($G$ after Griffith), which is the change of stored (elastic) energy per unit area change (for a list of the nomenclature used throughout this article, see table 1). Griffith was concerned with fracture in cracked, loaded solids and not cutting but the basic physics is the same, as noted by Lake & Yeoh [2]. This energy change is necessary to drive the fracture and provide the energy per unit area, $G_c$, required to create the new surface area and is a material property, the fracture toughness. The fracture criterion is thus

$$G \geq G_c$$

and is the basis of the discipline of fracture mechanics. Cutting can be regarded as a fracture mechanics process in which the input energy is provided, at least in part, by a cutting tool.

This energy-based criterion, equation (1.1), is a necessary, but not sufficient, condition for fracture. A local critical stress must also be achieved in order for fracture to occur. For sharp cracks in brittle solids, there is a high stress concentration and this condition is automatically satisfied as noted in [1]. This gives rise to linear elastic fracture mechanics, which embodies linear elastic behaviour and a single criterion, $G_c$, for fracture. For some biological materials, such as bone, this works well but for soft tissue large deformations result in crack blunting [3] and the second criterion, a critical stress, is necessary. A similar effect may occur with blunt tools though it should be noted that for soft solids a sharp tool may create a fracture without blunting [2], which can be advantageous in biological materials. In addition, it should be noted that large deformation around a tool may result in the tool contacting the fracture point. This is referred to here as ‘crack tip touching’ and can result in energy going into the fracture directly from the force on the tool and replaces, or adds to, $G_c$ as the driving force.

Modelling of cutting has a long history and was mostly confined to the machining of metals. In the early analyses, a fracture energy term was included in the analysis along with models of chip deformation and friction energy dissipation. In the 1950s [4], it was dropped because it was said that $G_c$ was the true surface work of the material, which is very small and could be ignored.
This is still the case in many machining modelling numerical codes but it was pointed out by Atkins [5] that the $G_c$ is not small and included the local plastic work as well as the surface energy necessary to create the surfaces. This is the same as the important contribution to fracture mechanics made by Orowan [6] and Irwin [7], who extended the Griffith analysis done for glass, where $G_c$ is indeed the surface work, to metals, where it is enhanced by local plasticity.

Figure 1 shows the model of cutting which will be used here to demonstrate the various mechanisms involved. It is a model of orthogonal cutting in which a surface layer, or chip, is removed by a wedge cutting with the edge normal to the cutting direction. Other geometries such as symmetrical blades or wires [8] are used but the basic scheme of the analyses is the same. In figure 1a, the chip is bent as the cutting tool moves forward. In all cases, a steady state is assumed in that the tool moves at a constant speed and the cut moves at the same speed. Cutting initiation is a more complex problem and will not be considered here. In general, the tool tip will not touch the fracture but can do for a finite opening and slope at the fracture point. For high values of the radius of curvature, $R$, as shown in figure 2, so that the bending is reversed when the chip passes the contact point and there is no energy dissipation. Indeed in these steady-state situations, there is never a change in the elastic energy but here we exclude any other dissipation which includes friction by putting $dU_f = 0$.

Equation (2.1) now becomes

$$F_c \, dx = bG_c \, dx,$$

where $G_c$ is the fracture energy per unit area.

This situation is employed in razor blade cutting [2] and with scissors to find the toughness of thin polymer films and is used for leaves [9]. Such materials do not usually yield and the test requires only the exclusion of friction. For very low modulus materials, the chip is a 'floppy offcut' as defined by Atkins [10], where the low stiffness, and hence low forces, gives crack tip touching and no friction. High stiffness is encountered in the splitting of wood with a wedge and where there is elastic energy which gives rise to an energy release rate and no touching, though friction is usually involved.
2.2. Elastic deformation \((dU_a = 0)\) with friction

Equation (2.1) is now

\[ F_c dx = b G_c dx + S dx, \]

and at the contact point

\[ \begin{align*}
S &= F_c \cos \theta - F_t \sin \theta \\
N &= F_c \sin \theta + F_t \cos \theta
\end{align*} \]

(2.3)

Thus,

\[ F_c dx = b G_c dx + (F_c \cos \theta - F_t \sin \theta) dx \]

and

\[ \frac{F_c}{b} (1 - \cos \theta) + \frac{F_t}{b} \sin \theta = G_c. \]

(2.4)

If \( \theta \) is known and \( F_c \) and \( F_t \) are measured then \( G_c \) can be found without defining the nature of the friction process. For \( S = 0 \), \( F_t = F_c (\cos \theta / \sin \theta) \) and equation (2.2) is retrieved.

Coulomb friction is often assumed and is given by

\[ S = \mu N, \]

where \( \mu \) is the friction coefficient, and from equation (2.3) we have

\[ F_t = F_c (\frac{\cos \theta - \mu \sin \theta}{\sin \theta + \mu \cos \theta}) \]

(2.5)

where \( \mu = \tan \beta \), and from equation (2.4)

\[ \frac{F_c}{b G_c} = \frac{1}{(1 - \cos \theta) + \sin \theta / \tan (\beta + \theta)}. \]

(2.6)

Note that \( F_t/F_c = 1/\mu \) at \( \theta = 0 \) and \( -\mu \) at \( 90^\circ \), i.e. a change of sign, and \( F_t = 0 \) for \( \theta = 1/\mu \). This has been suggested as a criterion to give smooth machined surfaces [11] and gives a minimum \( F_t/b \), i.e. when \( d/d(\theta) F_c b G_c = 0 \), and from equation (2.6) we have

\[ \left( \frac{F_c}{b G_c} \right)_{\min} = \frac{1}{1 - \cos \theta_o} = \frac{1}{1 - \sin \beta} \quad \text{and} \]

\[ \theta_o + \beta = 90^\circ. \]

(2.7)

This condition for a minimum in \( F_c \) is shown in figure 3 for \( \mu = 0.2 \), \( \beta = \tan^{-1}(0.2) = 11.3^\circ \), \( \theta_o = 79^\circ \) and \( (F_c/b G_c)_{\min} = 1.24 \).

The elastic bending of the chip may be described by simple bending theory in which the radius of curvature, \( R \), is related to the bending moment, \( M \), by

\[ \left( \frac{M}{R} \right) = \frac{E h^2}{12} \left( \frac{1}{R} \right). \]

(2.8)

where \( E \) is Young’s modulus and \( h \) is the chip thickness \((h_c = h\) here\). We may then plot the bending moment \( M/R \) versus \( 1/R \) as shown in figure 4, which also shows the linear relationship of equation (2.8). When the cut moves forward the moment goes from zero via the two loading lines shown to point A, where \( M = M_0 \) and \( R = R_c \). As the cut progresses, this point moves along the chip following the unloading line until it reaches the contact point, where \( M = 0 \) and \( 1/R = 0 \). The
elastic energy imparted to the chip on loading is the area under the curve, half of which is recovered while the remaining half (shaded area in figure 4) is the energy release rate $G = G_c$,

$$G_c = \frac{1}{2} \left( \frac{M_p}{b} \right) \left( \frac{1}{R_p} \right) = \frac{6}{Eh^3} \left( \frac{M_p}{b} \right)^2. \quad (2.9)$$

2.3. Elastic–plastic bending

The stress distribution for elastic bending of the chip is shown in figure 5a and is linear with the maximum of

$$\sigma = \frac{6}{b h^2} \left( \frac{M}{b} \right) \quad \text{and} \quad \frac{1}{R} = \frac{12}{E h^3} \left( \frac{M}{b} \right).$$

In figure 5b, this maximum stress reaches the tensile yield stress, $\sigma_y$, at the outer surface and

$$\sigma_y = \frac{6}{b h^2} \left( \frac{M_1}{b} \right) \quad \text{with} \quad \frac{1}{R} = \frac{1}{R_p} = \frac{12}{E h^3} \left( \frac{M_1}{b} \right) = \frac{2}{E} \left( \frac{\sigma_y}{h} \right).$$

Figure 5c shows the stress distribution for $M > M_1$, where the elastic–plastic interface is a distance $y = c$ from the central neutral axis. The upper limit for $M$ is shown in figure 5d when $c = 0$ and $M/b = M_p/b = \sigma_h h^2/4$. For $M_1/b < M_p/b$, the moment is given by

$$M = 2 \left( \int_0^c \sigma_y \frac{y^2}{c} dy + \int_c^{h/2} \sigma_y y dy \right),$$

i.e.

$$M = M_p \left( \frac{1 - \frac{2}{3} \left( \frac{R}{R_p} \right)^2}{\frac{R}{R_p}} \right). \quad (2.10)$$

At $y = c$, the strain is $\epsilon_y$ so that $\sigma_y = c/R$, and noting that $\epsilon_y = h/2R_p$ we have $2c/h = R/R_p$ and

$$M = M_p \left( \frac{1 - \frac{2}{3} \left( \frac{R}{R_p} \right)^2}{\frac{R}{R_p}} \right).$$

(2.10)

i.e. $M/b = M_p/b$ at $R = 0$ and $M/b = (M_p/b)(2/3) = M_1/b$ at $R = R_p$.

Figure 6 shows the bending moment radius of curvature relationship for this elastic–plastic case. The elastic line changes to equation (2.10) for $1/R \geq 1/R_p$ and the loading is shown in figure 4 and unloading is a line parallel to the elastic loading. The intercept on the $1/R$ axis, $1/R$, is the residual curvature of the chip. This upper shaded area is $G_c$, given by

$$G_c = \left( \frac{\sigma_h^2 h}{2E} \right) \left( 1 - \frac{2}{3} \left( \frac{R}{R_p} \right)^2 \right), \quad \frac{1}{R} \geq \frac{1}{R_p}, \quad (2.11)$$

and the lower shaded area is the energy dissipated in plastic work, i.e.

$$\frac{dU_d}{dx} = G_d \approx \left( \frac{\sigma_h^2 h}{2E} \right) \left( \frac{R_p}{R} \right) + \frac{1}{3} \left( \frac{R}{R_p} \right)^2 \frac{4}{3}, \quad (2.12)$$

and

$$\frac{R_p}{R} = \left( \frac{R_p}{R} \right) + \frac{1}{2} \left( \frac{R}{R_p} \right)^2 \frac{3}{2}. \quad (2.13)$$

It should be noted that, in equation (2.11), $G_c = (\sigma_h^2 h/2E)$ when $R \to 0$, i.e. $G_c$ is given by the elastic energy release rate when the elastic stress limit is $\sigma_y$. Thus, the parameter $\sigma_h^2 h/2EG_c = 1$ represents the maximum $G_c$ which can be achieved for a constant $\sigma_h$. For $\sigma_h^2 h/2EG_c < 1$, there will be no cutting since the required $G_c$ cannot be achieved. The limit for the onset of plastic bending is when $R/R_p = 1$, i.e. $G_d = 0$ and $\sigma_h^2 h/2EG_c = 3$. Equation (2.12) may be expressed as a function of $\sigma_h^2 h/2EG_c$ by substituting $R/R_p$. 

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**Figure 4.** Bending moment–radius of curvature diagram for elastic bending.

**Figure 5.** Stress states in elastic–plastic bending. (a) Elastic bending, (b) first yield, $\epsilon = h/2$, (c) elastic–plastic bending and (d) fully plastic bending, $\epsilon = 0$.

**Figure 6.** Bending moment–radius of curvature diagram for elastic–plastic bending.
from equation (2.11) and if we ignore friction since it is a constant factor times \( F_c/b \) in this case we have

\[
\frac{F_c}{bG_c} = 1 + \frac{1}{6} \left( \frac{x+1}{x-1} \right) (3 - x),
\]

where \( x = \sigma_0^2 h / 2EG_c \). This parameter is the ratio of \( h \) to a plastic zone length \( \tau = 2EG_c/\sigma_0^2 \).

### 2.4. Plastic shearing

A more efficient mode of plastic deformation is shown in figure 7, in which there is crack tip touching and the chip shears along a plane at an angle \( \phi \) and there is no bending. Invoking the Tresca yield criterion, the shear stress on the plane is \( \sigma_\gamma / 2 \). In this case the chip thickness, \( h_c \), is greater than \( h \) because of plane strain conditions and for finite friction

\[
\frac{h_c}{h} = \frac{\sin(\phi + \theta)}{\sin \phi}.
\]

For a tool movement of \( dx \) the movement along the tool–chip interface, \( dh_c \), is

\[
\frac{dh_c}{dx} = \frac{h_c}{h} = \frac{\sin(\phi + \theta)}{\sin \phi}.
\]

The shear displacement along the shear plane is

\[
\frac{dh_c}{dx} = \frac{\sin \theta}{\sin(\phi + \theta)}.
\]

and hence the energy dissipation terms may be derived

\[
\frac{dU_i}{dx} = S \frac{dx_c}{dx} = [(F_c - bG_c) \cos \theta - F_i \sin \theta] \frac{\sin \phi}{\sin(\phi + \theta)}
\]

and

\[
\frac{dU_\delta}{dx} = \frac{\sigma_\gamma b h}{2} \frac{dh_c}{dx} = \frac{\sigma_\gamma b h \sin \theta}{2} \frac{\sin \phi}{\sin(\phi + \theta)}.
\]

From equation (2.1), we now have

\[
\frac{F_c}{b} - G_c = \left[ \left( \frac{F_c}{b} - bG_c \right) \cos \theta - \frac{F_i}{b} \sin \theta \right] \frac{\sin \phi}{\sin(\phi + \theta)} + \frac{\sigma_\gamma h}{2} \frac{\sin \theta}{\sin(\phi + \theta)},
\]

i.e.

\[
\frac{F_c}{b} + \frac{F_i}{b} \tan \phi = G_c + \frac{\sigma_\gamma h}{2} \left( \tan \phi + \frac{1}{\tan \phi} \right).
\]

This solution forms the basis of a method for finding \( G_c \) from cutting [12] in which \( h \) is varied and \( h_c \) measured, hence giving \( \phi \). If \( F_c \) and \( F_i \) are measured then \( F_c/b + (F_i/b) \tan \phi \) may be plotted versus \( (h/2)(\tan \phi + 1/\tan \phi) \) to give \( G_c \) from the intercept and \( \sigma_\gamma \) from the slope of the straight line.

\[
\phi \text{ can be determined approximately by finding the value to minimize the external work done [13], as in the friction case discussed previously. Differentiating equation (2.18) with respect to } \tan \phi \text{ and equating to zero we have}
\]

\[
\frac{F_i}{b} = \frac{\sigma_\gamma h}{2} \left( 1 - \frac{1}{\tan^2 \phi_0} \right),
\]

where \( \phi_0 \) is the value of \( \phi \) at this minimum, and substituting in equation (2.18) we have

\[
\frac{F_c}{b} = G_c + \frac{\sigma_\gamma h}{\tan \phi_0},
\]

with \( 1/\tan^2 \phi_0 = 1 - (2/\sigma_\gamma h)(F_i/b) \).

In this case \( G_c \) and \( \sigma_\gamma \) may be found by fitting \( F_c \) and \( F_i \) data for a set of \( h \) values without the need for measuring \( h_c \). If Coulomb friction is assumed then, for this touching case, equation (2.5) becomes

\[
\frac{F_i}{b} = \left( \frac{F_c}{b} - G_c \right) \left( \frac{\cos \theta - \mu \sin \theta}{\sin \theta + \mu \cos \theta} \right).
\]

This enables \( \tan \phi_0 \) to be determined in terms of \( \mu \) and \( \theta \) using equations (2.16)–(2.18), and after some manipulation

\[
\frac{1}{\tan \phi_0} = \tan \left( \frac{\beta + \theta}{2} \right),
\]

where \( \tan \beta = \mu \), and equation (2.19) becomes

\[
\frac{F_c}{bG_c} = 1 - \frac{2}{c_\gamma} \frac{\tan \left( \frac{\beta + \theta}{2} \right) x}{x}.
\]

For \( \beta = 0 \), i.e. \( \mu = 0 \), this becomes

\[
\frac{F_c}{bG_c} = 1 - \frac{2}{c_\gamma} \frac{\tan \left( \frac{\theta}{2} \right) x}{x}.
\]
and may be compared with equation (2.17) for bending,
\[ F_c = \frac{G_m}{bG_c} = 1 + \frac{1}{6} \left( \frac{x + 1}{x - 1} \right) (\beta - x). \]

Figure 8 shows the three regimes of chip formation where \( F_c / bG_c \) is plotted versus \( x \). For \( x > 3 \), the deformation is elastic bending so the chips are straight, though such cutting is prone to cracking and unstable behaviour. For \( 1 < x < 3 \), there is elastic–plastic bending giving curled chips and smoother cutting behaviour. For \( 0 < x < 1 \), there is no bending and the chips are straight with a linear relationship between \( F_c / bG_c \) and \( h \). This regime is useful for finding \( G_c \) and usually gives smooth surfaces and stable forces. Equation (2.22) may be written as
\[ F_c = \frac{G_m}{bG_c} = 1 + \tan \left( \frac{\beta + \theta}{2} \right) \frac{h}{\delta_c}, \quad \frac{h}{\delta_c} < \frac{2}{\varepsilon'}. \]

where \( \delta_c = G_c / \sigma_y \) is the crack opening displacement and is a characteristic length for fracture.

3. Conclusion

The cutting process can be modelled as removing a chip using a wedge-shaped tool. The major parameter of the tool is the wedge angle \( \theta \) though sharpness, as defined by the tip radius, can have an effect but this is not considered here. The chip thickness is also a major factor in defining behaviour. The material properties which are relevant are Young’s modulus \( E \), the yield stress \( \sigma_y \) and the fracture toughness \( G_c \). Friction is also involved and can be described via the Coulomb friction coefficient \( \mu \). The parameter that drives the cutting is the cutting force \( F_c \), which gives rise to a transverse force normal to \( F_c \) and is \( F_t \).

Here \( F_c \) provides the external work for the cutting process which is dissipated as fracture \( G_c \), friction and plastic work via either elastic–plastic bending or shear. The former gives rise to chip curling and the latter to chip thickening. For elastic materials such as rubber, there is no plastic deformation so only fracture and friction are involved. Thin sheets give low forces so the latter can often be avoided. Leaf cutting tests using either blades or scissors are of this type.

Where there is plastic deformation then the type of behaviour can be defined by a non-dimensional thickness given by
\[ x = \frac{\sigma_y h}{2EG_c}, \]
and when \( x > 3 \) the facture is always elastic as observed when cutting thick chips which give cracking rather than forming smooth chips. For \( x < 3 \), plastically deformed curled chips from plastic bending occur. At small values of \( x \) plastic shearing occurs, which gives smooth cutting and thickened chips without curling. In this case, there is a linear correlation between cutting force and chip thickness which can be employed to determine material properties via cutting. The behaviour is quite complex and friction can influence the transition from shear to plastic bending.

It does appear that mechanics modelling and ideas derived from fracture mechanics enable the parameters involved in cutting to be quantified. In teeth, for example, rectangular specimens may be made and then layers of varying thickness removed with the forces \( F_c \) and \( F_t \) measured. From these readings, \( G_c \) and \( \sigma_y \) may be found which characterize both fracture and wear. Inhomogeneity may be accommodated by choosing the location where the cutting occurs. As previously mentioned, \( G_c \) may be found for many foods including leaves, fruit and meat and the interaction of teeth and food may be quantified to describe chewing efficiency and tooth wear. It is hoped that this will be useful in biological studies relating diet, teeth and evolution.

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