Bubble dynamics in a compressible liquid in contact with a rigid boundary

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A bubble initiated near a rigid boundary may be almost in contact with the boundary because of its expansion and migration to the boundary, where a thin layer of water forms between the bubble and the boundary thereafter. This phenomenon is modelled using the weakly compressible theory coupled with the boundary integral method. The wall effects are modelled using the imaging method. The numerical instabilities caused by the near contact of the bubble surface with the boundary are handled by removing a thin layer of water between them and joining the bubble surface with its image to the boundary. Our computations correlate well with experiments for both the first and second cycles of oscillation. The time history of the energy of a bubble system follows a step function, reducing rapidly and significantly because of emission of shock waves at inception of a bubble and at the end of collapse but remaining approximately constant for the rest of the time. The bubble starts being in near contact with the boundary during the first cycle of oscillation when the dimensionless stand-off distance \( g = s/R_m \), where \( s \) is the distance of the initial bubble centre from the boundary and \( R_m \) is the maximum bubble radius. This leads to (i) the direct impact of a high-speed liquid jet on the boundary once it penetrates through the bubble, (ii) the direct contact of the bubble at high temperature and high pressure with the boundary, and (iii) the direct impingement of shock waves on the boundary once emitted. These phenomena have clear potential to damage the boundary, which are believed to be part of the mechanisms of cavitation damage.

1. Introduction

The study of bubble dynamics in the neighbourhood of a rigid boundary is associated with cavitation erosion to propellers, turbines and pumps [1–6]. The cavitation damage mechanism is believed to be associated with shock waves [7,8] and bubble jetting [9], both of which are formed at the end of collapse. Similarly, the damage mechanism of an underwater explosion is associated with a shock wave emitted at the inception of an underwater explosion bubble and bubble jetting formed at the end of collapse [10–12].

Recent research on ultrasound cavitation bubbles is associated with several important medical applications, including extracorporeal shock wave lithotripsy [13–18], tissue ablation (histotripsy) [19–21], and oncology and cardiology [22]. In these applications, cavitation microbubbles absorb and concentrate a significant amount of energy from ultrasound, leading to shape oscillation, violent collapsing, shock waves and bubble jetting [23]. These mechanisms are also associated with sonochemistry [24,25] and ultrasound cavitation cleaning— one of the most effective cleaning processes for electrical and medical micro-devices [26,27].

A bubble initiated near a rigid boundary can be almost in contact with the boundary because of its expansion and migration to the boundary as a result of the attraction by the second Bjerknes force [28]. This leads to the direct impact of a liquid jet on the boundary once it penetrates through the bubble, the direct contact of the bubble at minimum volume at high pressure and high temperature with the boundary, and the direct impingement of shock waves on the boundary once emitted. We believe that these phenomena have clear potential
to damage the boundary. It is, therefore, very important to study bubble dynamics in near contact with a rigid boundary.

The boundary integral method (BIM) is grid-free in the flow domain and computationally efficient, and is thus widely used in the field of bubble dynamics. It has been applied for an axisymmetric configuration for a bubble near a rigid wall, a free surface or compliant surface [21,29–41] and for three-dimensional configurations [11,42–50]. The BIM model is suitable for an incompressible flow and does not account for the significant energy loss due to emission of shock waves associated with bubble dynamics. Lee et al. [51] modified the BIM model by removing empirically a part of the bubble potential energy at the end of the first cycle of oscillation.

The weakly compressible theory for spherical bubbles was developed by Prosperetti & Lezzi [52] and Lezzi & Prosperetti [53]. Wang & Blake [54,55] further developed this theory for non-spherical bubbles using the method of matched asymptotic expansions. In the weakly compressible theory, the flow to second order in the outer region far away from the bubble satisfies the wave equation and is obtained analytically. The flow to second order in the inner region near the bubble satisfies Laplace’s equation and thus is modelled using the BIM. Wang [56] modelled bubble dynamics near a rigid boundary in a compressible flow using the weakly compressible theory coupled with the BIM model for a bubble initiated at \( \gamma = s/R_m > 1 \). In this paper, we will study the closer cases for \( \gamma < 1 \), where a part of the bubble surface starts in near contact with the rigid boundary during the expansion phase. A thin layer of water is formed between the bubble surface and the boundary and remains thin subsequently. This leads to numerical instabilities in the BIM model. To avoid the numerical instabilities, we remove the thin layer of water between the bubble surface and the boundary, join the bubble surface with its image to the boundary, and simulate ‘the combined bubble’, following Ni et al. [57]. Our computations correlate well with the experiments of Philipp & Lauterborn [28] for both the first and second cycles of oscillation.

2. Weakly compressible theory

Consider the dynamics of a gas bubble near a rigid flat boundary in an inviscid and compressible liquid. A Cartesian-coordinate system is set as illustrated in Figure 1. The x-axis is on the rigid boundary and the z-axis is along the axis of symmetry for the configuration.

The maximum bubble radius \( R_m \) is chosen as the reference length, the density \( \rho_\infty \) in the undisturbed liquid is chosen as the reference density. The pressure reference is \( \Delta p = p_\infty - p \), where \( p_\infty \) is the pressure in the undisturbed liquid and \( p \) is the partial pressure of vapour of the bubble. The reference velocity is thus obtained as \( U = \sqrt{\Delta p/\rho_\infty} \). The reference time is obtained as \( T = R_m/U = R_m\sqrt{\rho_\infty/\Delta p} \), which provides the scale of the oscillation period of a bubble. As an example, the collapsing time required for a cavity collapsing from \( R = R_m \) to \( R = 0 \), obtained by Rayleigh, is \( T_{Rayleigh} = 0.915R_m\sqrt{\rho_\infty/\Delta p} \) [7].

We perform non-dimensionalization to the problem using the reference parameters and denote dimensionless quantities by subscript asterisks as follows:

\[
\hat{r} = \frac{r}{R_m},
\]

\[
\hat{t} = \frac{U}{R_m} t,
\]

\[
\hat{\varphi} = \frac{\varphi}{R_m U},
\]

\[
\hat{c}_s = \frac{c}{c_\infty},
\]

\[
\hat{p} = \frac{p - p_\infty}{\Delta p},
\]

where \( r = (x, y, z) \), \( t \) is the time, \( \varphi \) is the velocity potential of the liquid flow and \( p \) is the pressure. The speed of sound \( c \) is normalized by its value in the undisturbed liquid \( c_\infty \).

The compressibility of the liquid flow can be measured by the Mach number, defined in terms of the reference flow velocity \( U \) and the speed of sound \( c_\infty \) in the undisturbed liquid as follows:

\[
\varepsilon = \frac{U}{c_\infty},
\]

We assume that the Mach number \( \varepsilon \) is small, because the speed of sound in water is about 1500 m s\(^{-1}\), and the reference velocity \( U = \sqrt{\Delta p/\rho_\infty} \approx 10 \text{ m s}^{-1} \) at normal ambient pressure in water. The reference velocity \( U \) represents the mean radial velocity of oscillation. Transient compressible effects should be measured by the transient velocity of the bubble wall. The maximum bubble wall velocity is usually associated with the velocity of bubble jets, which is lower than 200 m s\(^{-1}\) at normal ambient pressure [58–67].

We divide the fluid domain into two asymptotic regions: the inner region near the bubble where \((x, y, z) = O(R_m)\) and the outer region far away from the bubble where \((x, y, z) = O(\Lambda)\), with \( \Lambda = c_\infty R_m/U \) being the wavelength of acoustic waves. Using the method of matched asymptotic expansions, the outer solution of the velocity potential to second order has been shown to satisfy the wave equation and is obtained analytically as follows [56,68]:

\[
\hat{\varphi} = \frac{V_t (t - \epsilon x)}{2\pi \epsilon} + O(\epsilon^2),
\]

where the overdot denotes the derivative in time. The outer solution is due to the acoustic radiation associated with the volume oscillation of the bubble.

The inner solution of the velocity potential \( \varphi \) to second order satisfies Laplace’s equation in the flow field and the kinematic and dynamic boundary conditions on the bubble surface \( S \) as follows:

\[
\nabla_s^2 \varphi = O(\epsilon^2),
\]

![Figure 1. Sketch map of bubble motion near a rigid boundary. The standoff distance between the initial bubble and rigid boundary is s, S is the bubble surface and the coordinates used are shown above.](http://rsfs.royalsocietypublishing.org/Downloaded from http://rsfs.royalsocietypublishing.org)
The far-field boundary condition of the inner solution is given as
\[
\varphi_w = \frac{1}{2\pi} \left( e V_s \left( t_s \right) - \frac{V_s \left( t_s \right)}{r_s} \right) + O(\varepsilon^2) \quad \text{as} \quad r_s \to \infty. \tag{2.5d}
\]

The initial condition on the bubble surface is given as
\[
\varphi_{w|t=0} = -R_p |t| \quad \text{on} \quad r_s = R_0, \tag{2.5e}
\]

where \( R_p \) is the initial radius of the bubble.

Examining the initial and boundary problem of (2.5), one can see that the compressible effects appear only in the far-field condition (2.5d) to the second-order approximation. As the basic equation is Laplace’s equation, this problem can be modelled using the BIM model. The details in the numerical model using the BIM model for the problem can be found in [70–72].

Bubble dynamics near a flat rigid boundary are modelled using the imaging method [39]. When the bubble surface is nearly in contact with the rigid boundary, simulations using the BIM model are often associated with numerical instabilities. To avoid the numerical instabilities, we remove a thin layer of liquid between the bubble surface and the boundary, join the bubble surface with its image to the boundary and simulate ‘the combined bubble’ [57]. In the simulations performed in this paper, the join takes place when the dimensionless minimum distance \( \delta_{\min} \) between the bubble surface and the boundary is in the range of 0.01–0.04.

A composite solution \( \varphi_c(\mathbf{r}, t) \) of the inner and outer solutions for the entire domain can be given as follows:
\[
\varphi_c(\mathbf{r}, t) = \varphi_w(\mathbf{r}, t) + \frac{1}{2\pi} \left( e V_s \left( t_s \right) - \frac{V_s \left( t_s \right)}{r_s} \right) + O(\varepsilon^2). \tag{2.6}
\]

The mechanical energy of a bubble system consists of the potential energy and the kinetic energy of the bubble system.

- The potential energy \( E_p \) is given as follows [23]:
  \[
  E_p = p_0 V_0 \left( \frac{V_0}{V_s} \right)^{\kappa-1} + \sigma A_s + V_s. \tag{2.7}
  \]

- The kinetic energy in the bubble gases is negligible because the density of gases is usually three orders of magnitude smaller than that of liquids.

- The local energy of a bubble system in a compressible liquid consists of the potential energy \( E_p \) and the local kinetic energy \( E_{L,k} \) as follows:
  \[
  E_{L,k} = E_p + E_{L,k} = \frac{p_0 V_0}{\kappa - 1} \left( \frac{V_0}{V_s} \right)^{\kappa-1} + \sigma A_s + V_s + 1 \int_{S_{LO}} \varphi \left. \varphi_{w,0} \right| dS. \tag{2.9}
  \]

3. Numerical analyses

The calculations are carried out for the dynamics of a laser-generated gas bubble having a maximum radius \( R_m = 1.45 \text{ mm} \) near a rigid boundary for the dimensionless standoff distances \( \gamma = s/R_m = 0.9, 0.6 \) and 0.3, to compare with the experimental data of Philipp & Lauterborn [28]. Other computational parameters are chosen as \( \kappa = 1.4, \varepsilon = 0.013, \sigma = 0.00051, R(0) = 0.1, R_s(0) = 31 \) and \( p_{\infty} = 127 \text{ [56]} \). The corresponding dimensional parameters are \( \rho = 1000 \text{ kg m}^{-3}, \sigma = 0.07 \text{ N m}^{-1}, p_0 = 98.1 \text{ kPa}, p_0 = 2.98 \text{ kPa}, R(0) = 1.45 \text{ mm}, R_s(0) = 307 \text{ ms}^{-1} \) and \( p_{\infty} = 12.1 \text{ kPa} \).
Figure 3. The motion of a bubble near a rigid boundary characterized by $\gamma = 0.9$, $K = 1.4$, $e = 0.013$, $\alpha = 0.00051$, $R_r(0) = 0.1$, $R_r(0) = 31.0$ and $\rho_{fr} = 127$, at various dimensionless times. The bubble shapes are during (a) the first expansion phase, (b and c) the first collapse phase in a singly- and doubly-connected form, respectively, (d) the second expansion phase and (e) the second collapse phase. (Online version in colour.)

Figure 3 shows the bubble shapes for $\gamma = 0.9$ at typical times. The bubble expands spherically but its lower surface is flattened by the rigid boundary towards the end of expansion (figure 3a). The bubble remains almost in contact with the boundary during collapse (figure 3b), as water cannot flow from directly below into the collapsing volume. Near the end of collapse, a liquid jet forms on the distal side of the bubble surface directed towards the boundary. Once it penetrates through the bubble, the jet impacts on the boundary immediately, which is associated with higher damage potential than with a jet formed away from the boundary. For the latter, the jet momentum reduces while it penetrates through the liquid before impact on the boundary.

After the jet penetrates through the bubble, a bubble ring is formed. The jet pointing to the boundary is redirected radially (away from the axis of symmetry) after it impacts on the boundary, which pushes the inner side of the bubble ring radially. As a result, the jet diameter increases, causing a compression of the bubble volume from inside. In the meantime, the bubble ring collapses from other sides rapidly except for the bottom, reaching its minimum volume in near contact with the boundary at $t_r = 2.30$ (figure 3c), when the bubble reaches the maximum pressure and temperature [4–6]. This is also associated with potential damage. In addition, a shock wave is emitted at the minimum bubble volume with high-pressure amplitude [4–6], it impacts on
the rigid boundary once it is emitted and has another clear damage potential.

The bubble ring then rebounds mainly upwards and externally along the boundary (figure 3d). It subsequently re-collapses mainly from the top and the external side (figure 3e). The radius of the bubble ring at the end of re-collapse is smaller than at the end of collapse. The bubble is kept almost in contact with the boundary during the second cycle of oscillation.

Figures 4 and 5 show the bubble shapes at various times for \( \gamma = 0.6 \) and 0.3, respectively. As expected, the lower part of the bubble surface starts being almost in contact with the boundary earlier than for \( \gamma = 0.9 \), at the middle and early stages of the expansion phase for \( \gamma = 0.6 \) and 0.3, as shown in figures 4a and 5a, respectively. In analogous to the case \( \gamma = 0.9 \), the lower part of the bubble surface keeps in near contact with the boundary subsequently and the liquid jet impacts the boundary once it penetrates through the bubble (figures 4b and 5b). Comparing figures 3b, 4b and 5b, one can see that the jet is sharper for a larger stand-off distance and its width decreases with the stand-off distance. More specifically, the ratio of the diameter of the middle cross section of a jet over its height (from the jet basement to its tip), as illustrated in figure 3b, at the end of collapse, decreases with the stand-off distance.

The jet is again redirected horizontally, pushing away the bubble from the inner side after it impacts on the boundary.

**Figure 4.** Bubble dynamics near a rigid boundary for \( \gamma = 0.6 \), at various dimensionless times, with the other parameters the same as in figure 3. The bubble shapes are during (a) the first expansion phase, (b and c) the first collapse phase in a singly- and doubly-connected form, respectively, (d) the second expansion phase and (e) the second collapse phase. (Online version in colour.)
The bubble ring collapses further from all sides except for the part almost in contact with the boundary, reaching its minimum volume and maximum pressure and temperature in near contact with the boundary at \( t^\ast = 2.32 \) and 2.34, respectively (figures 4c and 5c), when a shock wave is emitted and impinges on the boundary directly.

The bubble ring then further rebounds (figures 4d and 5d) and re-collapses (figures 4e and 5e), predominately from the top and external parts of the bubble surface. The radius of the bubble ring at the end of the second cycle of oscillation is again smaller than at the end of the first cycle. The maximum volume of the bubble during the second cycle increases as the bubble is initiated closer to the boundary.

We next consider the global behaviour of the bubble. Figure 6a shows the time history of the equivalent radius \( R_{eq} = \sqrt{3/(4\pi)}V_t \) of the bubble for the above three cases \( \gamma = 0.9, 0.6 \) and 0.3. The maximum radius reduces significantly from the first to the second cycle of oscillation, as does the oscillation period. The bubble maximum radius at the second cycle decreases to 0.56 \( R_m \), 0.59 \( R_m \), 0.65 \( R_m \) for \( \gamma = 0.9, 0.6 \) and 0.3, respectively, increasing as the bubble is initiated closer to the boundary.

Figure 6b shows the corresponding time history of the bubble centroid \( z_{cen}^\ast \). The bubble migrates slightly away from the boundary during expansion but migrates to the boundary significantly during collapse. The migration...
accelerates as the bubble is collapsing, reaching the maximum speed at the minimum volume. The bubble migrates to the boundary faster for a larger stand-off distance during the first cycle of oscillation for the three cases for $g_{0.9}$. This is contrary to the trend for $g_{0.3}$, where the bubble migrates to the boundary faster for a smaller stand-off distance. This is because the nearer part of the bubble surface is retarded by the boundary during the later stage of the expansion phase as $g_{0.9}$ and the retarding effects start earlier for a smaller stand-off distance $g$.

Figure 6c shows the history of the jet velocity $v_{jet}^*$ before jet impact, which increases rapidly with time before jet impact. It increases with the stand-off distance, taking the maximum values of $v_{jet}^* = 3.57$, 5.65 and 7.55, at $g = 0.3$, 0.6 and 0.9, respectively.

Figure 6d shows the history of the local energy $E_L/E_{L0}$ of the bubble system for the cases in figures 3, 4 and 5 where $E_{L0}$ is the initial local energy.

4. Comparison with experiments

Figure 7 shows the comparison of the bubble shapes obtained using the compressible BIM model and the experiment [28], for cavitation gas bubble dynamics near a rigid boundary for $R_m = 1.45$ mm and $g = 0.9$. The experimental and computational results are shown in the left and right columns, respectively. In addition, the computational results are overlapped with the experimental images for a direct comparison. The computation agrees very well with the experiment during the whole first cycle of oscillation (figure 7a). The expansion of the lower part of the bubble surface is retarded by the boundary at $t = 34$ μs. It approximately takes the shape of half of a sphere at its maximum volume at $t = 177$ μs, with the lower part of the bubble surface being flattened by the boundary. The upper part of the bubble surface then collapses down, assuming a cone shape at the middle stage of the collapse phase at $t = 296$ μs. The jet shown in the computational results is not visible in the experimental images due to opaqueness of the bubble surface. Nevertheless, the outer profiles of the bubble obtained
in the computation and experiment agree well. The bubble ring of the computation at the end of collapse at \( t = 353 \mu s \) agrees well with the experiment, when the bubble reaches its minimum volume.

Figure 7b shows the comparison during the second cycle of oscillation. The bubble surface in the experiment is not clear because of the physical instabilities that occurred. Nevertheless, the calculated bubble shapes correlate with the experiment data in terms of the outer profiles at various times. Both results show that the bubble rebounds and re-collapses nearly in contact with the boundary. They agree in terms of the external radius and height of the bubble ring.

A thin circular layer of water exists between the flat boundary and the lower part of the bubble surface as the later stage of the expansion phase, as the bubble is initiated with the stand-off distance being less than the maximum bubble radius. It becomes an annulus thin layer after the jet penetrates the bubble. The part of the bubble surface above the thin liquid layer is almost flat and the thickness of the liquid layer does not change significantly with time. This feature is shown in the images in figure 7 from \( t = 177 \) to \( 548 \) µs.

It can be estimated that the vertical acceleration \( a_z \) of the liquid in the thin layer is small,

\[
a_z = a_z \big|_{z=0} + O(\epsilon_{\text{min}}) = O(\epsilon_{\text{min}}),
\]

where \( \epsilon_{\text{min}} \) is the dimensionless maximum height of the thin layer of liquid, as \( a_z = 0 \) on the rigid boundary. From the \( z \)-component of the Euler equation, we have

\[
a_z = \frac{-\partial p_z}{\partial z}. \tag{4.2}
\]

The pressure in the gap can thus be estimated as follows:

\[
p_z = p_{zz} + O(\epsilon_{\text{min}}^2) = p_{zz} - \sigma_n \nabla \cdot n + O(\epsilon_{\text{min}}^2)
= p_{zz} + O(\epsilon_{\text{min}}^2) = p_{zz} + O(\epsilon_{\text{min}}^2). \tag{4.3}
\]
where the surface tension term is neglected as the curvature radius $\nabla \cdot \mathbf{n}$ is small on the flat part of the bubble surface.

The pressure in the thin layer of liquid between the bubble and the boundary is approximately constant and equal to the pressure of the bubble gas. The flow velocity within the thin layer must be close to zero. In addition the surface tension effects tend to keep the part of the bubble surface above the thin layer of liquid flat, as the pressure is constant and equal at its two sides.

Figure 8 shows the comparison between the computation and the experiment for $\gamma = 0.6$. The computation again agrees very well with the experiment during the whole first cycle of oscillation (figure 8a). The bubble takes the shape of a half sphere with the lower part being flattened by the wall at the middle stage of collapse at $t = 247 \mu s$. A bubble jet starts at $t = 285 \mu s$ and fully develops at $t = 302 \mu s$. The bubble ring calculated at the minimum volume at $t = 321 \mu s$ agrees well with the experiment. Figure 8b shows the comparison during the second cycle of oscillation. The bubble shapes calculated correlate with the experimental images. They agree well in terms of the radius and height of the bubble ring at the end of re-collapse and the period of the second cycle.

**Figure 8.** Comparison of the compressible BIM computation (in the right column) with the experiment (in the left column) [28] for the bubble shapes for $R_m = 1.45 \text{ mm}$ and $\gamma = 0.6$: (a) during the first cycle of oscillation and (b) during the second cycle of oscillation.

Figure 9 shows the comparison of the computation with the experiment for the bubble dynamics near a rigid boundary for $\gamma = 0.3$, starting from the late stage of collapse at $t = 280 \mu s$. The computation agrees very well with the experiment until the end of the collapse phase at $t = 354 \mu s$. The bubble ring calculated agrees well with the experiment during the early rebounding phase to $t = 389 \mu s$.

5. **Summary and conclusion**

Bubble dynamics in near contact with a rigid boundary are modelled using the weakly compressible theory coupled with the BIM. The near contact of the bubble surface with the boundary may cause numerical instabilities in the BIM model, which is avoided by joining the bubble surface with its image to the boundary. Our computations correlate well with the experimental data for both the first and second cycles of oscillation.

Some important features of the bubble dynamics near a rigid boundary have been identified.

1. A bubble initiated near a rigid boundary may be nearly in contact with the boundary because of its expansion and migration to the boundary, where a thin layer of water forms between the bubble and the boundary thereafter. The pressure in the thin layer of liquid is shown to be approximately constant and equal to the pressure of the bubble gas. The bubble side of the thin layer remains flattened because of surface tension effects. The flow velocity within the thin layer is close to zero.

2. The bubble starts nearly touching the rigid boundary during the expansion period when $\gamma \leq 1$, where $\gamma$ is
the dimensionless stand-off distance of the bubble from the boundary in terms of the equivalent maximum bubble radius. This leads to (i) the direct impact of a high-speed liquid jet on the boundary once it penetrates through the bubble at the end of collapse, (ii) the direct contact of the bubble ring at high temperature and high pressure at its minimum volume with the boundary, and (iii) the direct impingement of a shock wave on the boundary once emitted at the end of collapse. These phenomena have clear potential to damage the boundary. We believe these three phenomena are possible new mechanisms of cavitation damage.

(3) As observed in [28,56], the bubble starts touching the boundary during the second cycle of oscillation when $1 \lesssim \gamma \lesssim 2$. This leads to (i) the direct impact of a high-speed liquid jet on the boundary once it penetrates through the bubble at the end of re-collapse, (ii) the direct contact of a bubble ring at high temperature and high pressure at its second minimum volume with the boundary, and (iii) the direct impingement of a shock wave on the boundary once emitted at the end of re-collapse.

(4) It has been observed that the jet is sharper and thinner at a larger stand-off distance.

(5) The time history of the energy of the bubble system follows a step function, reducing significantly because of emissions of shock waves at its inception and the end of collapse but remaining constant during the rest of the time. The loss of the local energy at the end of collapse increases with the stand-off distance.

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